Optics Design and Performance of an Ultra-Low Emittance Damping Ring for the Compact Linear Collider

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Abstract

A high-energy (0.5–3.0 TeV centre of mass) electron-positron Compact Linear Collider (CLIC) is being studied at CERN as a new physics facility. The design study has been optimized for 3 TeV centre-of-mass energy. Intense bunches injected into the main linac must have unprecedentedly small emittances to achieve the design luminosity $10^{35}$ cm$^{-2}$s$^{-1}$ required for the physics experiments. The positron and electron bunch trains will be provided by the CLIC injection complex.

This thesis describes an optics design and performance of a positron damping ring developed for producing such ultra-low emittance beam.

The linear optics of the CLIC damping ring is optimized by taking into account the combined action of radiation damping, quantum excitation and intrabeam scattering. The required beam emittance is obtained by using a TME (Theoretical Minimum Emittance) lattice with compact arcs and short period wiggler magnets located in dispersion-free regions. The damping ring beam energy is chosen as 2.42 GeV. The lattice features small values of the optical functions, a large number of compact TME cells, and a large number of wiggler magnets. Strong sextupole magnets are needed for the chromatic correction which introduces significant nonlinearities, decreasing the dynamic aperture. The nonlinear optimization of the lattice is described. An appropriate scheme of chromaticity correction is determined that gives reasonable dynamic aperture and zero chromaticity. The nonlinearities induced by the short period wiggler magnets and their influence on the beam dynamics are also studied. In addition, approaches for absorption of synchrotron radiation power produced by the wigglers are discussed.

Realistic misalignments of magnets and monitors increase the equilibrium emittance. The sensitivity of the CLIC damping ring to various kinds of alignment errors is studied. Without any correction, fairly small vertical misalignments of the quadrupoles and, in particular, the sextupoles, introduce unacceptable distortions of the closed orbit as well as intolerable spurious vertical dispersion and coupling due to the strong focusing optics of the damping ring. A sophisticated beam-based correction scheme has been developed in order to bring the design target emittances and the dynamic aperture back to the ideal value. The correction using dipolar correctors and several skew quadrupole correctors allows a minimization of the closed-orbit distortion, the cross-talk between vertical and horizontal closed orbits, the residual vertical dispersion and the betatron tune coupling.

The small emittance, short bunch length, and high current in the CLIC damping ring could give rise to collective effects which degrade the quality of the extracted beam. A number of possible instabilities and an estimate of their impact on the ring performance are briefly surveyed. The effects considered include fast beam-ion instability, coherent synchrotron radiation, Touschek scattering, intrabeam scattering, resistive-wall wake fields, and electron cloud.

Keywords
damping ring, intra-beam scattering, ultra-low emittance, wiggler, dynamic aperture
Un collisionneur linéaire électron-positron compact, nommé CLIC, est à l’étude au CERN. Il devra permettre des expériences à des énergies comprises entre 0.5 et 3 TeV dans le centre de masse. L’étude est optimisée pour 3 TeV. Pour atteindre la luminosité nominale de $10^{35} \text{ cm}^{-2}\text{s}^{-1}$, les paquets d’électrons devront avoir une intensité élevée et une émittance d’un petitie sans précédent.

Ce document présente la conception de l’optique et les performances d’un anneau d’amortissement étudié pour atteindre les émittances requises.


Le défauts d’alignements des aimants et des moniteurs de position augmentent l’émittance d’équilibre et nous étudierons la sensibilité de l’anneau CLIC d’amortissement par rapport à ces défauts. Sans correction, de faibles erreurs d’alignement des quadrupoles, et plus encore des hexapôles introduisent des erreurs d’orbite fermée trop importantes ainsi qu’une dispersion parasite verticale et un couplage lié à la forte focalisation qui sont bien au delà du seuil de tolérance. Un schema de correction lié à la mesure du faisceau et qui préserve l’émittance nominale et l’ouverture dynamique est proposé. L’usage combiné d’aimant dipolaires et quadrupolaires d’azimuth non-nul permettront de minimiser les défauts d’orbite, la dispersion verticale et le couplage betatronique global.

La combinaison d’une émittance faible, de paquets courts et d’un courant de faisceau total élevé peut induire des effects collectifs qui dégradent la qualité du faisceau extrait de l’anneau. Un bref inventaire d’instabilités potentielles et de leur impact sur la performance de l’anneau est présenté. Les effets considérés ici sont l’instabilité rapide ion-faisceau, la radiation synchrotronique cohérente, la diffusion Touscheck, la diffusion interne aux paquets, les champs de sillage liés à la résistivité de la paroi de la chamber à vide, et la présence de nuages d’électrons.

**Mots clés**

anneau d’amortissement, diffusion intra-beam, émittance, ondulateur, ouverture dynamique
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Chapter 1

General introduction

1.1 Prospects for high energy physics

The Standard Model is a highly successful theory, agreeing perfectly with all confirmed data from particle accelerator experiments, and describing accurately the characteristics of three of the four fundamental forces, the electromagnetic force and the strong and weak nuclear forces. But the Standard Model has its limitations. As a theory, it is not entirely satisfactory, incorporating many arbitrary parameters. Moreover, it tells us nothing about gravity, the fourth and weakest of the fundamental forces; and there are hints from non-accelerator experiments observing neutrinos—ghostly particles that barely interact with other matter—that their behaviour cannot be fully accounted for in the Standard Model.

As at other accelerator laboratories, the top priorities at European Organization for Nuclear Research (CERN) will be experiments probing beyond the Standard Model [1, 2]. Indeed, this is surely the only responsible motivation for major new accelerators.

There are good reasons to expect a wealth of new physics in the TeV range, in particular that connected with the origin of particle masses. This new physics might include an elementary Higgs boson, but most physicists would expect the new physics to be more complex, perhaps including new spectroscopy of supersymmetric particles or other excitations.

The first exploration of the TeV energy range will be made with the Large Hadron Collider (LHC) [3, 4]. The LHC is presently under construction at CERN and scheduled for completion in 2007. The LHC will collide protons at a 14 TeV centre-of-mass energy (up to about 1 TeV in collisions between complex, multi-quark particles; not all of the energy is available for creating new particles).

It is expected that high-energy $e^+e^-$ colliders will be needed to help unravel the TeV physics, to be unveiled by the LHC. An electron–positron collider with centre-of-mass energies between 0.5 TeV and 1 TeV would be able to explore in detail the properties of any relatively light Higgs boson and have a chance of producing lighter supersymmetric particles, but would probably not be able to explore all the supersymmetric spectrum, nor study in detail any new strong interactions.

A multi-TeV linear $e^+e^-$ collider (with centre-of-mass energies between 0.5 TeV and 3 TeV or even higher) would be able to distinguish smaller extra dimensions than a sub-TeV
machine [5]. This is the objective of the Compact Linear Collider, or CLIC as it is known.

1.2 Overview of the CLIC complex

A high luminosity electron-positron Compact Linear Collider [6, 7] has been under study for several years at CERN in the framework of an international collaboration of laboratories and institutes aimed at providing the HEP community with a new accelerator-based facility for the post-LHC era. A new scheme of beam acceleration enabling electron-positron collisions at energies between 0.2 TeV (the final energy of the LEP collider) up to a maximum of about 5 TeV, realized in steps, was proposed by the CLIC study team.

The CLIC design parameters have been optimized for a nominal centre-of-mass energy of 3 TeV with a luminosity of about $10^{35} \text{cm}^{-2}\text{s}^{-1}$, but the CLIC concept allows its construction to be staged without major modifications. The possible implementation of a lower-energy phase for physics would depend on the physics requirements at the time of construction. In principle, a first CLIC stage [8] could cover centre-of-mass energies between $\sim 0.2$ and 0.5 TeV with a luminosity of $\mathcal{L} = 10^{33} - 10^{34} \text{cm}^{-2}\text{s}^{-1}$, providing an interesting physics overlap with the LHC. This stage could then be extended first to 1 TeV, with $\mathcal{L}$ above $10^{34} \text{cm}^{-2}\text{s}^{-1}$, and then to multi-TeV operation, with $e^+e^-$ collisions at 3 TeV, which should break new physics ground. A final stage might reach a collision energy of 5 TeV or more.

In order to achieve high energies with a linear collider, a cost-effective technology is of prime importance. In conventional linear accelerators, the RF power used to accelerate the main beam is generated by klystrons. To achieve multi-TeV energies, high accelerating gradients are necessary to limit the lengths of the two main linacs and hence the cost. Such high gradients are easier to achieve at higher RF frequencies since, for a given gradient, the peak power in the accelerating structure is smaller than at low frequencies. For this reason, a frequency of 30 GHz has been chosen for CLIC so as to attain a gradient of 150 MV/m. However, the production of highly efficient klystrons is very difficult at high frequency. Even for X-band at 11.5 GHz, a very ambitious programme has been necessary at SLAC and KEK to develop prototypes that come close to the required performance. At even higher frequencies, the difficulties of building efficient high-power klystrons are significantly larger.

Instead, the CLIC design is based on the two-beam accelerator scheme. The sketch of Fig. 1.1 shows the overall layout of the CLIC complex. The RF power is extracted from a low-energy high-current drive beam, which is decelerated in power-extraction transfer structures (PETSs) of low impedance. This power is then directly transferred into the high-impedance structures of the main linac and used to accelerate the high-energy low-current main beam, which is later brought into collision. In other words, in this method the RF power for a section of the main linac is extracted from a secondary, low-energy, high-intensity electron beam running parallel to the main linac. The two-beam approach offers a solution that avoids the use of a large number of active RF elements, e.g. klystrons or modulators, in the main linac. This potentially eliminates the need for a second tunnel. The total length of the two linacs required for the nominal energy of 3 TeV is $\sim 28$ km. Two interaction points (IPs) are foreseen, one for $e^+e^-$ and one for $\gamma\gamma$ interactions.

In the CLIC scheme, the drive beam is created and accelerated at low frequency (0.937 GHz) where efficient klystrons can be realized more easily. The pulse current and intensity of the beam is then increased in a frequency-multiplication chain consisting of one delay loop and two combiner rings. This drive-beam generation system can be installed at a central site,
thus allowing easy access and replacement of the active RF elements. A new facility CTF3 [9] is being built at CERN to demonstrate the technical feasibility of the key concepts of the novel CLIC RF power source.

![Figure 1.1: Schematic overall layout of the CLIC complex.](image)

The two-beam acceleration method of CLIC ensures that the design remains essentially independent of the final energy for all the major subsystems, such as the main beam injectors, the damping rings, the drive-beam generators\(^1\), the RF power source, the main-linac and drive-beam decelerator units, as well as the beam delivery systems (BDSs). The main tunnel houses both linacs, the drive-beam lines, and the BDSs.

The general layout of the main-beam injection complex is illustrated in Fig. 1.2. The polarized electrons are obtained from a laser-driven DC gun, and the primary electrons for positron production from a laser-driven 1.875 GHz RF gun. The electron and positron beams are accelerated to 2.42 GeV in stages by a 1.875 GHz injector linac (see Fig. 1.2). This linac accelerates alternately the train of electrons and the train of positrons. A DC dipole magnet inflects the \(e^-\) beam and the \(e^+\) beam in a main electron damping ring and in a positron pre-damping ring, respectively. It also allows the beam to be sent towards a dump where some beam instrumentation will be implemented. From the pre-damping ring, the \(e^+\) beam is injected into the main positron damping ring. After the damping ring, the beam is accelerated to 9 GeV and longitudinally compressed in a two-stage bunch length compressor.

\(^1\)The only difference between the drive-beam generation schemes for high and low colliding-beam energies is the length of the modulator pulse (the installed hardware is exactly the same).
The bunches injected into the CLIC main linac must have unprecedentedly small emittances to achieve the design luminosity required for the physics experiments. The luminosity $L$ in a linear collider can be expressed as a function of the effective transverse beam sizes $\sigma_{x,y}$ at the interaction point:

$$L = H_D \frac{N_{bp}^2}{4\pi \sigma_x \sigma_y} N_{bt} f_{rr}.$$  \hspace{1cm} (1.1)

Here, the bunch population is denoted by $N_{bp}$, the number of bunches per beam pulse by $N_{bt}$, the number of pulses per second by $f_{rr}$, and the luminosity enhancement factor by $H_D$. The factor $H_D$ is usually in the range of 1-2, and it describes the increase in luminosity due to the beam–beam interaction, which focuses the $e^+e^-$ beams during collision.

The above parameters are strongly coupled. An important example of a coupled parameter is the bunch length $\sigma_s$. In a given main linac the bunch length is a function of the bunch population, larger $N_{bp}$ requiring larger $\sigma_s$. In turn, the optimum ratio $N_{bp}/\sigma_s$ for the beam-beam collision is limited by beamstrahlung.

In order to achieve a small vertical beam size at the IP, the vertical phase space occupied by the beam – the vertical emittance $\epsilon_{y}$ – must be small. The total effective beam size at the IP can be expressed in a simplified way as a function of the total emittance and the focal strength of the final-focus system:

$$\sigma_{y,\text{eff}} \propto \sqrt{\beta_y^* (\epsilon_{y,DR} + \Delta \epsilon_{y,BC} + \Delta \epsilon_{y,\text{linac}} + \Delta \epsilon_{y,\text{BDS}})}.$$  \hspace{1cm} (1.2)

where $\beta_y^*$ is the vertical betatron function at the interaction point. First, a beam with an ultra-low emittance $\epsilon_{y,DR}$ must be created in the damping ring. The target value of the vertical normalized emittance $\gamma \epsilon_{y,DR}$ (where $\gamma$ is the Lorentz factor) for the electron and positron main CLIC damping rings is $\gamma \epsilon_{y,DR} \leq 3$ nm. The total emittance growth $\gamma (\Delta \epsilon_{y,BC} + \Delta \epsilon_{y,\text{linac}} + \Delta \epsilon_{y,\text{BDS}})$ is due to the following number of challenges: longitudinal compression and subsequent transportation to the main linac $- \Delta \epsilon_{y,BC}$, acceleration in the main linac $- \Delta \epsilon_{\text{linac}}$, and finally, collimation and strong focusing in the BDS $- \Delta \epsilon_{\text{BDS}}$. The total vertical emittance growth should not exceed 10 nm for the nominal $\beta_y^* = 90$ μm to achieve the design luminosity. The total horizontal emittance growth is mainly due to the final focus system, collimation system, and bunch compressors.

Figure 1.2: Main beam injector layout.
1.3 CLIC damping rings

The CLIC damping rings serve as the particle sources for the CLIC linear collider. The laser-driven DC gun (electron source) and laser-driven 1.875 GHz RF gun (source of primary electrons for subsequent positron production) cannot provide the desired extremely small transverse beam emittances. Therefore, the electron and positron beams generated by a conventional gun and positron target, respectively, must be stored in damping rings to obtain the target ultra-low beam emittances by virtue of the synchrotron radiation. Positron or electron bunch trains, which consist of 220 bunches separated by 16 cm, have to be extracted from the positron or electron damping ring at the repetition rate of 150 Hz. The design bunch population is $2.56 \times 10^9$ particles. For both electron and positron main damping rings, the target values of the normalized transverse emittances $\gamma \epsilon_{x,y}$ for the extracted $e^-$ and $e^+$ beams are 450 nm horizontally and 3 nm vertically. Each of these values is about an order of magnitude smaller than the present world record emittances achieved at the KEK-ATF prototype damping ring. Moreover, for CLIC the longitudinal beam emittance at extraction should not exceed 5000 eVm in order to satisfy the requirements for the subsequent bunch compressor.

Usually, positron generation from a primary electron beam results in positron bunches with large emittances. The expected upper limit for both horizontal and vertical normalized emittances is $\gamma \epsilon_{x,y} < 50 000 \, \mu$m. To decouple the wide aperture required for the incoming positron beam from the final emittance requirements of the main linac, an $e^+$ pre-damping ring with a large dynamic acceptance and relatively large equilibrium emittances is needed. In the case of electron production, taking into account the smaller incoming normalized emittance of 7 $\mu$m provided by the high brilliance injector linac, a single damping ring similar to the main positron damping ring will be sufficient.

1.4 Scope of the thesis

The subject of the thesis work is to design the optics and to optimize the performance of the positron main damping ring for the CLIC. The work described in this PhD thesis was performed in the framework of the CLIC study group. Chapter 2 describes basic theoretical principles of radiation damping and quantum excitation, including equilibrium beam properties for different high-brilliance lattice types. Chapter 3 is devoted to the effect of intra-beam scattering that has a strong impact on the beam emittances in the CLIC damping ring. Chapter 4 presents the lattice design for the CLIC damping ring. Chapter 5 describes a non-linear optimization of the damping ring lattice in order to increase its dynamic aperture. In Chapter 6, the nonlinearities induced by a NdFeB permanent wiggler optimized for the damping ring and of their influence on the beam dynamics are studied. This chapter also includes a section devoted to the absorption of synchrotron radiation power. In Chapter 7, the sensitivity to different alignment errors and the emittance recovery achieved by correcting the closed orbit distortion, the residual vertical dispersion and the betatron coupling are studied. Chapter 8 surveys a number of possible instabilities and estimates their impact on the ring performance. Chapter 9 summarizes the conclusions of this study. The design of the $e^+$ pre-damping ring is not part of the thesis theme.
Chapter 2

Basic beam optics

2.1 The first order equations of motion and Twiss parameters

The charged particle motion in a circular accelerator is described by the general equation

\[ \dot{\mathbf{v}} = \frac{e}{\gamma m_0} \mathbf{v} \times \mathbf{B} \quad (2.1) \]

where \( \gamma = \frac{1}{\sqrt{1 - v^2/c^2}} \) is the Lorentz factor. For guiding charged particle beams along the design orbit (reference orbit), bending forces are needed. Only transverse magnetic field is considered since the electric field is not efficient for bending the trajectory of a relativistic particle with \( v \approx c \). For example, a magnetic field of 1 T gives the same bending force as an electric field of 300 MV per meter for a relativistic particle. Most particles of the beam deviate slightly from the design orbit. In order to keep these deviations small at all times, focusing forces are required.

In order to describe particle trajectories in the vicinity of the reference orbit, we introduce a right-handed Cartesian co-ordinate system \( \{\mathbf{\hat{y}}, \mathbf{\hat{x}}, \mathbf{\hat{s}}\} \) as shown in Fig. 2.1 where \( \rho \) is the bending radius produced by the bending magnet with a dipole magnetic field in the vertical direction. If an ultrarelativistic electron with momentum \( p_0 \) passes through the vertical homogeneous field \( B_0 \) generated by the dipole magnet with a flat pole shape, the bending radius \( \rho \) of its trajectory is given by

\[ \frac{1}{\rho} [\text{m}^{-1}] = \frac{eB_0}{p_0} = 0.2998 \frac{B_0 [\text{T}]}{E [\text{GeV}]} \quad (2.2) \]

since the Lorentz force is equal to the centrifugal force. In Eq. (2.2), \( E \) and \( e \) are the particle energy and charge of the electron, respectively.
Figure 2.1: Co-ordinate system \{\vec{y}, \vec{x}, \vec{s}\} used to describe particle trajectories in the vicinity of the reference orbit.

The focusing (defocusing) force is provided by quadrupole magnets which have four iron poles shaped in the form of a hyperbola \(xy = R_0^2/2\). The field of the quadrupole is zero on the \(s\)-axis but it increases linearly with the distance from \(s\)-axis:

\[
B_y = gx, \quad B_x = gy \quad \text{where} \quad g = \frac{\partial B_y}{\partial x} = \frac{\partial B_x}{\partial y} = \frac{2\mu_0 NI}{R_0^2}.
\]

Here, \(N\) is the number of turns of wire in the coil, \(I\) is the current in the wire. For a positively charged particle, the quadrupole with \(\partial B_y/\partial x < 0\) is horizontally focusing and vertically defocusing. This quadrupole will become horizontally defocusing and vertically focusing if the current direction or the particle charge or the direction of the particle motion is reversed. The strength of focusing is characterized by the normalized gradient \(K_1\),

\[
K_1 = \frac{e}{\rho_0} \frac{\partial B_y}{\partial x} = \frac{1}{B_0\rho} \frac{\partial B_x}{\partial x}.
\]

Note that \(K_1\) is positive for horizontally focusing quadrupole and negative for the vertically focusing quadrupole.

Many of the older alternating synchrotrons like the CERN proton synchrotron PS or the DESY electron synchrotron have been built with so-call "combined function" bending magnets, i.e. magnets which combine a dipole field for deflection and a quadrupole field for focusing. The strength of focusing for such magnets can be characterized by \(K_1\) or the field index \(n\) which have the following relation

\[
n = \frac{\rho}{B_0} \frac{\partial B_y}{\partial x} = \rho^2 K_1
\]

The new accelerators and storage rings are usually equipped with "separated function" magnets, i.e. dipoles for deflection and quadrupole magnets for focusing. However, the
combined function bending magnets are still used in some modern machines, for example, at the ATF damping ring [12] in KEK. The field gradient of each ATF bending magnet (total number of bending magnets in the ATF damping ring is 36 units, with $B_0 = 0.9$ T and $\rho = 5.73$ m) is equal to $6.122$ T/m, which gives $K_1 = 1.187$ m$^{-2}$ or $n = 38.98$.

For a circular machine consisting of bending and quadrupole magnets only, the first order equations of motion are given by

\[
\frac{d^2 x}{ds^2} - K_1(s)x = \frac{1}{\rho(s)} \frac{\Delta p}{p_0}, \tag{2.5}
\]
\[
\frac{d^2 y}{ds^2} + K_1(s)y = 0. \tag{2.6}
\]

These are basic equations for the particle trajectory $x(s)$, $y(s)$ in linear approximation when the particle has a momentum $p_0 \pm \Delta p$ (off-momentum particle). A momentum $p_0$ is called the design (reference) momentum $p_0$. In the following, we will use relative momentum deviations $\delta = \Delta p/p_0$. Equations (2.5–2.6) define the so-called “linear optics” of the machine.

The functions $\rho(s)$ and $K_1(s)$ are periodic functions of $s$ with a period that is equal to the circumference of the closed orbit of the circular machine. The general solution of Eq. (2.5) is the sum of the complete solution of the homogeneous equation (when Eq. (2.5) is equal to zero) and a particular solution of the inhomogeneous equation $D'' - K_1(s)D = 1/\rho(s)$. In this case, the transverse particle motion can be separated into two parts:

\[
x = x_\beta + D_x \delta \quad y = y_\beta
\]

where

- $D_x \delta$ – characterizes the first order energy dependence of the closed orbit. The horizontal periodic dispersion function $D_x$ describes the deviation of the closed orbit for off-momentum particles with momentum offset $\Delta p$ from the reference orbit (orbit for particle with momentum $p_0$),
- $x_\beta$ – describes the betatron oscillation around this closed orbit.

In matrix form, the solution to Eqs. (2.5–2.6) can be expressed as

\[
\begin{pmatrix}
  x \\
  x'
\end{pmatrix}_s =
\begin{pmatrix}
  C_x & S_x \\
  C'_x & S'_x
\end{pmatrix}\begin{pmatrix}
  x \\
  x'
\end{pmatrix}_{s_0} + \delta \begin{pmatrix}
  D \\
  D'
\end{pmatrix} \tag{2.7}
\]

Here, a prime denotes the derivative with respect to $s$, $x(s_0)$ and $x'(s_0)$ are the initial values, $C_x(s)$ and $S_x(s)$ are two periodic linear independent solutions of the homogeneous equation which satisfy the following condition $CS' - C'S = 1$.

In the ideal machine (without any betatron coupling, only vertical dipole fields and neither misalignments nor field errors), the vertical dispersion $D_y$ is zero. The vertical motion $y(s)$, $y'(s)$ is characterized by the functions $C_y(s) \neq C_x(s)$ and $S_y(s) \neq S_x(s)$.

Firstly let us consider betatron part of motion. The functions $C(s)$ and $S(s)$ can be written in terms of the Twiss parameters $\beta(s)$, $\alpha(s)$ and $\gamma(s)$ introduced by Courant and Snyder [10]. The Twiss parameters are related to each other and the betatron phase $\phi(s)$ by

\[
\phi_{x,y}(s) = \int \frac{1}{\beta_{x,y}(s)} ds, \quad \alpha_{x,y}(s) = \frac{1}{2} \beta'_{x,y}(s), \quad \gamma_{x,y}(s) = \frac{1 + \alpha_{x,y}^2(s)}{\beta_{x,y}(s)}. \tag{2.8}
\]
\( \beta(s), \alpha(s) \) and \( \gamma(s) \) and dispersion functions satisfy the periodic boundary conditions

\[
\begin{align*}
\alpha_x, \alpha_y(s) &= \alpha_x, \alpha_y(s + C), \\
\beta_x, \beta_y(s) &= \beta_x, \beta_y(s + C), \\
\gamma_x, \gamma_y(s) &= \gamma_x, \gamma_y(s + C), \\
D_{x, y}(s) &= D_{x, y}(s + C).
\end{align*}
\]  

(2.9)

The horizontal and vertical betatron tunes of the machine, \( Q_x \) and \( Q_y \) have the following values

\[
\begin{align*}
Q_x &= \frac{1}{2\pi} \int_0^C \frac{1}{\beta_x(s)} ds, \\
Q_y &= \frac{1}{2\pi} \int_0^C \frac{1}{\beta_y(s)} ds.
\end{align*}
\]  

(2.10)

Here, \( C \) is the circumference of the machine. Often also the betatron phase advance between two points \( s_1 \) and \( s_2 \) is expressed as a fraction of \( 2\pi \), i.e. \( \nu_{x, y} = \phi_{x, y}(s_1 \rightarrow s_2)/2\pi \).

The transformation matrix from \( s_0 \) to \( s \) in Eq. (2.7) is given by

\[
\begin{pmatrix}
C(s) & S(s) \\
C'(s) & S'(s)
\end{pmatrix} = 
\begin{pmatrix}
\sqrt{\frac{\nu}{\beta_0}}(\cos \phi + \alpha_0 \sin \phi) & \sqrt{\beta_0 \beta_0} \sin \phi \\
\frac{1}{\sqrt{\beta_0}}((\alpha_0 - \alpha) \cos \phi - (1 + \alpha \alpha_0) \sin \phi) & \sqrt{\frac{\nu}{\beta_0}}(\cos \phi - \alpha \sin \phi)
\end{pmatrix}
\]  

(2.11)

with

\[
\Delta \phi = \phi(s) - \phi(s_0)
\]

The functions \( \{C_x(s), C'_x(s), S_x(s), S'_x(s)\} \) correspond to the \( \{\alpha_x(s), \beta_x(s), \gamma_x(s)\} \) and \( \{C_y(s), C'_y(s), S_y(s), S'_y(s)\} \) to \( \{\alpha_y(s), \beta_y(s), \gamma_y(s)\} \).

The betatron phase advance \( \Delta \phi_{s_1 \rightarrow s_2} \) between \( s_1 \) and \( s_2 \) can be found as

\[
\cos \Delta \phi_{s_1 \rightarrow s_2} = \frac{1}{2} \text{trace matrix (2.11)} \bigg|_{s_1 \rightarrow s_2} = \frac{C(s_1 \rightarrow s_2) + S'(s_1 \rightarrow s_2)}{2}
\]  

(2.12)

The Twiss parameters can be found by the following linear transformation

\[
\begin{pmatrix}
\beta \\
\alpha \\
\gamma
\end{pmatrix} = 
\begin{pmatrix}
C^2 & -2CS & S^2 \\
-C'C & SC' + S'C & -SS' \\
C'^2 & -2S'C & S'^2
\end{pmatrix}
\begin{pmatrix}
\beta_0 \\
\alpha_0 \\
\gamma_0
\end{pmatrix}
\]  

(2.13)

The periodic horizontal dispersion \( D_x \) and its derivative \( D'_x \) can be expressed in term of \( C_x(s), S_x(s) \):

\[
D_x(s) = \left(1 - S'(s)\right)D_x(s) + S(s)D'_x(s), \quad D'_x(s) = \frac{C'(s)D_x(s) + (1 - C(s))D'_x(s)}{4\sin^2 \pi Q_x}
\]  

(2.14)
where

\[
\tilde{D}_x(s) = S(s) \int_{s_0}^{s} \frac{1}{\rho(t)} C(t) \, dt - C(s) \int_{s_0}^{s} \frac{1}{\rho(t)} S(t) \, dt
\]  

(2.15)

\[
\tilde{D}'_x(s) = S'(s) \int_{s_0}^{s} \frac{1}{\rho(t)} C(t) \, dt - C'(s) \int_{s_0}^{s} \frac{1}{\rho(t)} S(t) \, dt
\]

Taking functions \(C_x(s), C'_x(s), S_x(s)\) and \(S'_x(s)\) from Eq. (2.11) from \(s_0\) to \(s_0 + C\) (C - ring circumference), the horizontal periodic dispersion is finally expressed as

\[
D_x(s) = \frac{\sqrt{\beta(s)}}{2 \sin \pi Q} \int \frac{\sqrt{\beta(t)}}{\rho(t)} \cos(|\phi(t) - \phi(s)| - \pi Q) \, dt
\]  

(2.16)

### 2.2 Horizontal emittance

The general solution of Eq. (2.5) for the on-momentum particle \(\delta = \Delta p/p_0 = 0\) can be written as

\[
x(s) = \sqrt{\epsilon_{x0}} \sqrt{\beta_x(s)} \cos(\phi_x(s) - \phi_{x0})
\]  

(2.17)

\[
x'(s) = -\frac{\sqrt{\epsilon_{x0}}}{\sqrt{\beta_x}} (\sin(\phi(s) - \phi_0) + \alpha_x(s) \cos(\phi(s) - \phi_0))
\]  

(2.18)

The integration constant \(\phi_0\) is determined by the initial conditions. The particle co-ordinates \(\{x(s), x'(s)\}\) given by Eqs. (2.17–2.18) satisfy the following equality at any \(s\)

\[
A_x = \gamma_x(s)x^2(s) + 2\alpha_x(s)x(s)x'(s) + \beta_x(s)x'^2(s) = \text{const} \quad \text{at any } s
\]  

(2.19)

The constant quantity \(A_x\) is called Courant-Snyder invariant [10]. It is easy to see that Eq. (2.19) is the representation of an ellipse in the \(\{x, x'\}\) plane. Since the ellipse is determined by the Twiss parameters, the shape and orientation of the ellipse will change along the orbit, but the ellipse area, which is equal to \(\pi \epsilon\), will be constant. The mean value of \(A_x\) over all particles in the beam, \(\epsilon_{x0} \equiv \langle A_x \rangle\), is called the horizontal natural (or geometrical) equilibrium emittance of the beam. The horizontal beam size depends on \(\epsilon_{x0}\). In a non-dispersive place of machine where \(D_x = 0\), the rms horizontal beam size is equal to \(\sqrt{\epsilon_{x0} \beta_x}\).

Ignoring current-dependent effect, the natural horizontal equilibrium emittance of a flat beam generated by synchrotron radiation in a ring is [11]

\[
\epsilon_{x0} = \frac{C_q \gamma^2 I_5}{J_x I_2}
\]  

(2.20)

where \(C_q = (55\hbar)/(32\sqrt{3}mc) = 3.84 \times 10^{-13}\) m (for electrons or positrons). The parameters \(I_5\) and \(I_2\) are the fourth and fifth synchrotron radiation integrals (synchrotron integrals),
respectively. $J_x$ is the horizontal partition number. Sometimes in the literature the definition of the normalized equilibrium emittance is used. It refers to the value $\gamma \epsilon_{x0}$.

Another important quantity, $H_x$, is called the dispersion invariant or lattice invariant. It is defined as

$$H_x(s) = \gamma_x D_x^2 + 2\alpha_x D_x D'_x + \beta_x D_x^2 = \frac{1}{\beta_x} \left[ D_x^2 + \left( \beta_x D'_x - \frac{1}{2} \beta'_x D_x \right)^2 \right] \quad (2.21)$$

In the ideal machine the vertical dispersion invariant $H_y(s)$ is zero since $D_y = 0$ everywhere. But in the presence of betatron coupling or horizontal dipole field or some alignment errors, $H_y(s)$ can have a significant value. The synchrotron radiation integrals are defined [11] as

$$I_1 = \oint \frac{D_x}{\rho} \, ds \qquad I_2 = \oint \frac{1}{\rho^2} \, ds \qquad I_3 = \oint \frac{1}{|\rho^3|} \, ds$$

$$I_4 = \oint \frac{D_x}{\rho} \left( \frac{1}{\rho^2} - 2K_1 \right) \, ds = \oint \frac{(1-2n)D_x}{\rho^3} \, ds \qquad I_5 = \oint \frac{H_x}{|\rho^3|} \, ds \quad (2.22)$$

Here, $n$ is the field index of bending magnets. The horizontal equilibrium emittance is proportional to $H_x$ via $I_5$. The damping partition numbers are defined as

$$J_x = 1 - \frac{I_4}{I_2} \quad J_y = 1 \quad J_\varepsilon = 2 + \frac{I_4}{I_2} \quad (2.23)$$

### 2.3 Vertical emittance

In an ideal uncoupled ring there is no vertical dispersion or linear coupling. The photons are not emitted exactly in the direction of the particle motion but at small opening angle. In this case, the minimum vertical emittance is determined by the vertical opening angle of the synchrotron radiation and it has the value [13]:

$$\epsilon_{y0,\min} = \frac{13 \, C_q \, \oint \frac{\beta_y}{|\rho^0|} \, ds}{55 \, J_y \, \oint \frac{1}{\rho^2} \, ds} \quad , (2.24)$$

which is negligible even for the CLIC parameters. When the damping time $\tau$ is dominated by wigglers, the equation (2.24) can be approximated as $\epsilon_{y0,\min} = 0.9 \times 10^{-13} \beta_y / \rho_y^w$ (m · rad).

In the following, the vertical and horizontal zero-current emittances (i.e., no effect of IBS) will be denoted as $\epsilon_{y0}$ and $\epsilon_{x0}$. The contribution to the vertical zero-current emittance from the vertical dispersion, that usually results from alignment errors (transverse displacements, roll angles and so on) of dipole, quadrupole and sextupole magnets is given by

$$\epsilon_{y0,d} = \frac{J_x}{J_y} \langle H_y \rangle \sigma^2 \approx 2J_x \frac{\langle D_y^2 \rangle}{J_y} \beta_y \sigma^2 \quad (2.25)$$

where $\sigma^2 = \langle (\Delta p)^2 \rangle / \rho_0^2$ is the square of the rms relative momentum deviation. We assume that the vertical dispersion along the ring is a spurious dispersion, such as might be expected after a dispersion correction has been performed.

The increase of the vertical emittance due to weak betatron coupling that can arise, for example, from skew quadrupole components of the field can be expressed as

$$\epsilon_{y0,\beta} = \kappa \epsilon_{x0} \quad (2.26)$$
where $\kappa$ is the coupling factor. In Chapter 7 the emittance growth due to various alignment errors will be studied in detail.

In the presence of both vertical dispersion and betatron coupling, the vertical emittance in the limit of zero bunch charge (i.e., the emittance due to synchrotron radiation and quantum excitation only) is the sum

$$\epsilon_{y0} = \epsilon_{y0,\text{min}} + \epsilon_{y0,d} + \kappa \epsilon_{x0}.$$  \hspace{1cm} (2.27)

### 2.4 Radiation damping

Positrons (electrons) lose energy by synchrotron radiation which results in a reduction of both transverse and longitudinal components of the momentum. To compensate for the energy loss, accelerating field in RF cavities are used but only the longitudinal component of the momentum is restored. The lost transverse momentum is not compensated. This leads to steady reduction of the transverse betatron oscillation or to damping.

- **Energy loss due to synchrotron radiation**

Charged particles radiate when they are deflected in the magnetic field. Photons are emitted along the tangent to the particle trajectory. Integrating the synchrotron power $P_{SR}$ around the machine we obtain $U_0$ – the energy loss per turn

$$U_0 = \oint P_{SR} dt = \oint C_\gamma \frac{c E^4}{\rho^2} ds = C_\gamma \frac{E^4}{\rho}$$  \hspace{1cm} (2.28)

where $c$ is the velocity of light and

$$P_{SR} = \frac{c C_\gamma E^4}{2\pi \rho^2} = \frac{2}{137 \cdot 3} \frac{hc \gamma^4}{\rho^2}, \quad C_\gamma = \frac{4\pi}{3} \frac{r_0}{(m_ec^2)^3} = 8.858 \times 10^{-5} \left[ \text{m V}^{-3} \right]$$  \hspace{1cm} (2.29)

$h_0 = 197 \times 10^{-15}$ MeV·m, $r_0 = 2.82 \times 10^{-15}$ m is the classical radius of electron.

- **Damping times**

For the general case, where focusing and bending may occur in the same magnet, the transverse ($\tau_x$, $\tau_y$) and longitudinal damping times ($\tau_p$) are given as

$$\tau_x = \frac{2E_0T_0}{J_x U_0} = \frac{2E_0T_0}{(1 - I_4/I_2)U_0} = \frac{2E_0T_0}{1 - \oint \frac{\rho ds}{\rho^2} \left( \frac{1}{\rho^2} + 2K_1 \right) ds} U_0$$  \hspace{1cm} (2.30)

$$\tau_y = \frac{2E_0T_0}{J_y U_0} = \frac{2E_0T_0}{U_0}$$  \hspace{1cm} (2.31)

$$\tau_p = \frac{2E_0T_0}{J_z U_0} = \frac{2E_0T_0}{(2 + I_4/I_2)U_0} = \frac{2E_0T_0}{1 + \oint \frac{\rho ds}{\rho^2} \left( \frac{1}{\rho^2} + 2K_1 \right) ds} U_0$$  \hspace{1cm} (2.32)
where $T_0$ is the revolution time of particles along the orbit of the machine. For a separated function lattice where the focusing and bending functions are performed by different magnets the damping times simplify to

$$
\tau_x = \tau_y = \frac{2E_0T_0}{U_0} = \frac{3T_0}{r_0^\gamma \gamma^3 I_2}, \quad \tau_p = \frac{\tau_{x,y}}{2} = \frac{E_0T_0}{U_0} = \frac{3}{2} \frac{T_0}{r_0^\gamma \gamma^3 I_2}
$$

(2.33)

Here, $\gamma$ is the Lorentz factor. In this case, $J_x \simeq 1$, $J_y = 1$ and $J_\varepsilon \simeq 2$ since the contribution from $I_4/I_2$ is usually $\sim 10^{-3}$.

The vertical damping partition number is $J_y = 1$ for any lattice. The sum of the damping partition numbers for the three planes is a constant:

$$J_x + J_y + J_\varepsilon = 4$$

This result is known as the Robinson Theorem [14].

Due to the radiation damping, the transverse beam emittances $\varepsilon_x$, $\varepsilon_y$ and rms energy deviation (spread) $\sigma_\varepsilon \equiv \sigma_p$ evolve with time according the following equations:

$$
\frac{d\varepsilon_x}{dt} = -2 \frac{\varepsilon_x}{\tau_x}, \quad \frac{d\varepsilon_y}{dt} = -2 \frac{\varepsilon_y}{\tau_y}, \quad \frac{d\sigma_\varepsilon^2}{dt} = -2 \frac{\sigma_\varepsilon^2}{\tau_p}
$$

(2.34)

However, the final values of $\varepsilon_x$, $\varepsilon_y$ and $\sigma_\varepsilon$ are not zero since the process of quantum excitation occurs. The balance between the radiation damping and quantum excitation results in equilibrium values of the beam emittance and energy spread that will be discussed further below.

- **Synchrotron oscillation**

A synchronous particle gains an amount of energy from the RF cavities which is equal to its energy loss per turn

$$U_{rf} = e\hat{V}_{rf} \cos(2\pi f_{rf}t + \varphi_0) = U_0$$

where $\hat{V}_{rf}$ is the amplitude of RF voltage, $f_{rf}$ is the RF frequency, $\varphi_0 < \pi/2$ is the synchronous RF phase angle that corresponds to the synchronous particle ($t_0$). A particle with a positive energy deviation $\varepsilon = E - E_0$ moves on a larger orbit length with respect to the synchronous particle and therefore arrives later ($t_0 + \Delta t$) at the RF cavity. Such particle gains less energy from RF which reduces its energy deviation. Conversely, a particle with a negative energy deviation $-\varepsilon$ goes on a shorter orbit and receives higher energy gain from the RF. Thus, the dependence of the particle displacement $l(t) = c\Delta t$ from the synchronous particle (center of beam) on energy deviation can be expressed as

$$\frac{1}{c} \frac{dl(t)}{dt} = \alpha_p \frac{\varepsilon}{E_0}
$$

(2.35)

where the momentum compaction factor $\alpha_p$ is

$$\alpha_p = \frac{1}{C_0} \int \frac{D_x}{\rho} ds = \frac{I_1}{C_0}
$$

(2.36)

The second-order differential equation for the evolution of the particle energy deviation in time is

$$
\frac{d^2\varepsilon}{dt^2} + \frac{2}{\tau_p} \frac{d\varepsilon}{dt} + \Omega_s^2 \varepsilon = 0
$$

(2.37)
Assuming that the damping rate $1/\tau_p$ is small with respect to the oscillation frequency $\Omega_s$, the solution of Eq. (2.37) can be found as [15]

$$\varepsilon(t) = A e^{-t/\tau_p} \cos(\Omega_s t - \theta) \quad (2.38)$$

where the synchrotron angular frequency $\Omega_s$ is

$$\Omega_s = \frac{c}{C} \sqrt{\frac{2\pi \hbar \alpha_p e V_{rf} |\cos \varphi_0|}{E_0}} \quad (2.39)$$

Here, $h = C f_{rf} / c = C / \lambda_{rf}$ is the harmonic number. Inserting Eq. (2.38) into Eq. (2.35) and taking the integral in $t$, the damped harmonic oscillator equation describing the evolution of the longitudinal position is found as

$$l(t) = \frac{\alpha c}{E_0 \Omega_s} A e^{-t/\tau_p} \sin(\Omega_s t - \theta) \quad (2.40)$$

The longitudinal motion can be represented in the phase space of two conjugated variables $\{\varepsilon, l\}$ as an ellipse. The rms relative energy spread $\sigma_\delta$ and rms bunch length $\sigma_s$ (for zero current) are related as

$$\sigma_s = \sigma_\delta C \sqrt{\frac{\alpha_p E_0}{2\pi \hbar e V_{rf} |\cos \varphi_0|}} = \sigma_\delta C \sqrt{\frac{\alpha_p E_0}{2\pi \hbar (e V_{rf}^2 - U_0^2)^{1/2}}} \quad (2.41)$$

As one can see from Eqs. (2.34), (2.38) and (2.40) the transverse emittances, relative energy spread and rms bunch length appear to be damped to zero value. However, this is not realistic, since we must include in our consideration another important mechanism - quantum excitation which produces random excitation of betatron and synchrotron oscillations.

### 2.5 Quantum excitation and equilibrium beam properties

- **Equilibrium emittance**

The equilibrium emittances and energy spread are determined by the balancing of the radiation damping and quantum excitation. Each particle performs betatron oscillation about its equilibrium orbit. If a particle emits a photon at a place with non-zero dispersion, it loses energy and instantly starts performing betatron oscillations about a different equilibrium orbit. Synchrotron radiation produces random excitation of betatron and synchrotron oscillations. As a consequence the betatron amplitudes change and the statistical nature of the emission of photons leads to a continuous increase of the betatron amplitudes and of the beam size. This together with the damping effect leads to an equilibrium beam emittance.

The emission of one photon of energy $u = \hbar \omega$ at a point with non-zero dispersion gives rise to a change in the off-energy orbit, and hence introduces a change in the betatron motion,

$$\delta x_\beta = -D_x u / E_0, \quad \delta x'_\beta = -D'_x u / E_0$$
that according to Eq. (2.19) induces an increase of the horizontal invariant by
\[
\delta A_x = \gamma_x \delta (x'_x) + 2 \alpha_x \delta (x'_x) + \beta_x \delta (x'_x^2) = \frac{1}{E_0^2} u^2 \mathcal{H}_x(s)
\]

With \( \mathcal{N} \) being the number of photons emitted per unit time and averaging over all particles in the beam, the quantum excitation of the beam emittance per unit time is
\[
\frac{d\epsilon_x}{dt} = \frac{\langle N \langle u^2 \rangle \mathcal{H}_x \rangle}{2E_0^2}
\]
where the photon flux \( \mathcal{N} \langle u^2 \rangle \) is given by [16]
\[
\mathcal{N} \langle u^2 \rangle = \frac{15\sqrt{3} P_{SR}}{8 u_c} \cdot \frac{11}{27} u_c^2 = \frac{55}{24\sqrt{3}} u_c P_{SR} \quad \text{where} \quad u_c = \hbar \omega_c = \frac{3 \hbar c \gamma^3}{2 \rho} \quad (2.42)
\]
Here, \( u_c \) is the characteristic photon energy. Including the radiation damping term from Eq. (2.34), we therefore have the following total rate of change of \( \epsilon_x \)
\[
\frac{d\epsilon_x}{dt} = -2 \frac{\epsilon_x}{\tau_x} + \frac{\langle N \langle u^2 \rangle \mathcal{H}_x \rangle}{2E_0^2} \quad (2.43)
\]
The damping time \( \tau_x \) can be expressed in terms of synchrotron radiation as \( \tau_x = 2E_0 / J_x \langle P_{SR} \rangle \).
Solving Eq. (2.43) for the condition \( d\epsilon_x/dt = 0 \) and taking the definition of photon flux from Eq. (2.42) and SR power from Eq. (2.29), we deduce the equilibrium horizontal emittance \( \epsilon_{x0} \)
\[
\epsilon_{x0} = \frac{\tau_x}{4E_0} \langle N \langle u^2 \rangle \mathcal{H}_x \rangle = \frac{55}{32\sqrt{3} mc} \frac{\hbar \gamma^2}{J_x \langle P_{SR} \rangle} = C_q \gamma^2 \frac{\langle \mathcal{H}_x \rangle}{\langle \rho^2 \rangle} = C_q \gamma^2 I_5 \quad (2.44)
\]
This equation is the same as that defined by Eq. (2.20) in Sec. 2.2.

**Equilibrium relative energy deviation**

Using Eq. (2.38), we can write an expression for the evolution of energy deviation \( \epsilon \) in the presence of radiation damping and quantum excitation as the sum over all the previous photon emissions
\[
\epsilon(t) = \sum_{i, t > t_i} u_i \exp \left[ -\frac{(t - t_i)}{\tau_p} \right] \cos \Omega_s(t - t_i)
\]
It follows that the mean-square standard deviation of energy is therefore
\[
\langle \sigma_{\epsilon}^2 \rangle = \sum_i \frac{\langle u^2 \rangle}{2} \exp \left[ -2 \frac{(t - t_i)}{\tau_p} \right] = \frac{\langle N \langle u^2 \rangle \rangle}{4} \int \exp \left[ -2 \frac{(t - t_i)}{\tau_p} \right] dt_i = \frac{\langle N \langle u^2 \rangle \rangle \tau_p}{4}
\]
Inserting photon flux definition from Eq. (2.42) and using representation of longitudinal damping time as \( \tau_p = 2E_0 / J_x \langle P_{SR} \rangle \), we obtain the relative energy spread \( \sigma_{\epsilon} = \sigma_{x0}/E_0 \) or relative momentum deviation \( \sigma_p / p_0 \):
\[
\sigma_{\epsilon}^2 = \left( \frac{\sigma_{x0}}{E_0} \right)^2 = \left( \frac{\sigma_p}{p_0} \right)^2 = \frac{55 \hbar \gamma^2}{32\sqrt{3} mc J_x \langle P_{SR} \rangle} = C_q \gamma^2 \frac{\langle \mathcal{H}_x \rangle}{\langle \rho^2 \rangle} = C_q \gamma^2 I_3 = C_q \gamma^2 J_x I_2 = C_q \gamma^2 J_x (\rho / I_2) \quad (2.45)
\]
• Equilibrium bunch length

Equilibrium bunch length follows from the relation with energy deviation given by Eq. (2.41)

\[
\sigma_{s0} = \sigma_c C \left( \frac{\alpha_0 E_0}{2\pi h e \tilde{V}_r \cos \varphi_0} \right) = C \left( \frac{E_1 \alpha_p}{J_x E_0 h \left[ (e \tilde{V}_r / U_0)^2 - 1 \right]^{1/2}} \right)
\]

where \( E_1 = 2.639 \times 10^6 \) eV.

• R.M.S. beam size

In the non-dispersive \((D_x = 0)\) region of the accelerator, the rms horizontal beam size \(\sigma_x\) and divergence \(\sigma_x'\) for a Gaussian distribution of the particles in the beam is defined only by the betatron oscillation as

\[
\sigma_x = \sqrt{\epsilon_x \beta_x(s)} \quad \sigma_x' = \sqrt{\epsilon_x \gamma_x(s)}
\]

In the regions of accelerator with finite dispersion, the total horizontal beam size and divergence include also a contribution from the energy spread, namely

\[
\sigma_x = \left[ \epsilon_x \beta_x(s) + D_x^2(s) \left( \frac{\sigma_{x0}}{E_0} \right)^2 \right]^{1/2}
\]

\[
\sigma_x' = \left[ \epsilon_x \gamma_x(s) + D_x^2(s) \left( \frac{\sigma_{x0}}{E_0} \right)^2 \right]^{1/2}
\]

2.6 The minimum emittance

Taking into account IBS, we have to find a lattice which will produce a beam with ultra-low emittance. For an isomagnetic guide field \((\rho_0 = \text{constant in magnets, } \rho = \infty \text{ elsewhere})\), the horizontal emittance become

\[
\gamma \epsilon_{x0} = \frac{C_q \gamma^3 \langle H_x \rangle_{\text{mag}}}{J_x \rho_0} = \frac{C_q \gamma^3 \frac{1}{L} \int_0^L H_x(s)}{J_x \rho_0}
\]

where the integral of the horizontal dispersion invariant is taken only along the one bending magnets.

The transformation of the horizontal lattice functions through a non-focusing \((K_1 = 0)\) sector bending magnet (see Appendix A) with length \(L\) and small bending angle \(\theta \ll 1\) is given by

\[
\beta(s) = \beta_0 - 2\alpha_0 s + \gamma_0 s^2
\]

\[
\alpha(s) = \alpha_0 - \gamma_0 s
\]

\[
\gamma(s) = \gamma_0
\]

\[
D(s) = D_0 + D'_0 s + \rho_0 (1 - \cos \theta)
\]

\[
D'(s) = D'_0 + \sin \theta
\]

16
where the index "0" refers to the entrance of the bending magnet. The deduction of Eq. (2.50) and transfer matrices of the most important magnets are given in the Appendix A. Knowing the optical functions at the entrance of the bending magnet (referred by the index 0), integral of \( H_x \) through the magnet can be analytically developed up to second order in \( L/\rho \):

\[
I_{mag} = \int_0^L H_x ds = \left( \gamma D_0^2 + 2\alpha D_0 D_0' + \beta D_0'^2 \right) L + \left( \alpha_0 D_0 + \beta_0 D_0' \right) \frac{L^2}{\rho} - \left( \gamma_0 D_0 + \alpha_0 D_0' \right) \frac{L^3}{3\rho} + \left( \frac{\beta_0}{3} - \frac{\alpha_0 L}{4} + \frac{\gamma_0 L^2}{20} \right) \frac{L^3}{\rho^2} \tag{2.51}
\]

If the horizontal lattice functions \( \beta^*, \alpha^*, \gamma^*, D^* \) and \( D^* \) are known at the center of the bending magnet, then the integral \( I \) can be represented as

\[
I = \left( \gamma^* D^* + 2\alpha^* D^* D^* + \beta^* D^* \right) L - \left( \gamma^* D^* + \alpha^* D^* \right) \frac{L^3}{12\rho} + \left( \frac{\beta^*}{12} + \frac{\gamma^* L^2}{320} \right) \frac{L^3}{\rho^2} \tag{2.52}
\]

The approximations (2.51–2.52) are valid for most light sources because for a bending angle \( \theta < 20^\circ \) the error is < 1%.

In most types of lattice structure, which are developed for modern synchrotron machines, there are two basic layouts shown in Fig. 2.2:

Figure 2.2: Two basic situations of dispersion behavior in the bending magnets for most periodic lattice structures: A) \( D_0 = 0 \) and \( D_0' = 0 \) at the entrance of the bending magnet, B) \( D^* = 0 \) and \( \alpha^* = 0 \) at the middle of the bending magnet.

A The beam enters or comes out of the bending magnet with zero dispersion and zero slope of dispersion, \( D_0 = 0 \) and \( D_0' = 0 \) at the entrance of the bending magnet.

B The horizontal dispersion \( D_x \) and betatron function \( \beta_x \) have optical symmetry with respect to the bend center, i.e. \( D_x^* = 0 \) and \( \alpha_x^* = 0 \) at the middle of the bending magnet.
2.6.1 Symmetry with respect to the bend center

In the case B, the integral (2.52) reaches a minimum if the lattice functions at the middle of the bending magnet have the following values:

\[ \beta_m^* = \frac{L}{2\sqrt{15}} \quad D_m^* = \frac{L\theta}{24} \quad \alpha_m^* = 0 \quad D_m^* = 0 \]  
(2.53)

This set of optical functions gives the theoretical minimum emittance [17, 18, 19] (TME):

\[ \gamma \epsilon_{x0m} = \frac{C_q \gamma^3}{J_x} \frac{\theta^3}{12\sqrt{15}}. \]  
(2.54)

Although the minimum emittance could be decreased by using a combined function bending magnets with defocusing gradient \(-K_1\) (in this case \(J_x\) becomes \(>1\) see Eqs. (2.22–2.23)), the emittance decrease is quite small unless the gradients are very large. Moreover if the damping rate is dominated by wiggler magnets, the change of \(J_x\) due to defocusing gradient in bending magnets is very small. Usually, a combined function bending magnet makes sense only to the extent that it helps the matching of lattice functions in the TME cells.

If the \(\beta^*\) and \(D^*\) at the middle of the bend are different from the optimum values of Eq. (2.53), but the symmetry is still preserved, i.e. \(D^{*'} = 0, \alpha^* = 0\), the resulting equilibrium emittance \(\epsilon_{x0}\) will be larger than the optimum one \(\epsilon_{x0m}\). The detuning factor \(\epsilon_r \equiv \epsilon_{x0}/\epsilon_{x0m}\), is expressed as a function of the relative optical functions \(\beta_r \equiv \beta^*/\beta_m^*\) and \(D_r \equiv D^*/D_m^*\) [20]

\[ \epsilon_r = \frac{5}{8} \frac{D_r}{\beta_r} [D_r - 2] + \frac{9}{2} \left[ \frac{1}{4\beta_r} + \frac{\beta_r}{9} \right] \]  
(2.55)

The average dispersion in the detuned lattice is usually larger than the dispersion for the non-detuned lattice. This implies a reduction in the strengths of the quadrupoles and the chromatic correction sextupoles and potentially an increase in the ring’s dynamic aperture.

The family of curves for different values of detuning factor \(\epsilon_r\) ranging from 1 to 7 in the \(\beta_r, D_r\) diagram is shown in Fig. 2.3. The dispersion is maximum for a fixed value of the emittance detuning factor \(\epsilon_r\), when the relative horizontal beta function is equal to \(\beta_r = \epsilon_r\). In this case, the relative maximum dispersion \(D_{r,max}\) and the emittance are

\[ D_{r,max} = 1 + \frac{2}{\sqrt{5}} \sqrt{\epsilon_r^2 - 1} \quad \gamma \epsilon_{x0} = \epsilon_r \frac{C_q \gamma^3}{J_x} \frac{\theta^3}{12\sqrt{15}}, \quad \text{at} \quad \beta_r = \epsilon_r \]  
(2.56)

The horizontal damping partition \(J_x\) for the detuned lattice becomes the following

\[ J_x \approx 1 - (1 + K_1 \rho^2) \theta^2 / 6. \]  
(2.57)

If the optical symmetry at the middle of the bend is broken, i.e, \(D^{*'} \neq 0\) and \(\alpha^* \neq 0\), the relation between \(\beta_r\) and \(D_r\) for the constant detuning factor \(\epsilon_r\) is described by an equation of second order curves:

\[ a_\beta_r^2 + 2b_\beta_r D_r + cD_r^2 + 2d_\beta_r + 2vD_r = \epsilon_r \]  
(2.58)
Figure 2.3: Relative dispersion $D_r$ versus relative beta function $\beta_r$ for constant emittance detuning factors.

Figure 2.4: The family of the second order curves at constant $\epsilon_r = 1.8$ for different values of $D^{\ast\prime}$ and $\alpha^{\ast}$. 
where \( a, b, c, d, v \) are the functions of \( D^{*'} \) and \( \alpha^* \). The family of the second order curves at constant \( \epsilon_r = 1.8 \) for the different values of \( D^{*'} \) and \( \alpha^* \) is shown in Fig. 2.4. The ellipse for \( D^{*'} > 0 \) at fixed \( \alpha^* \) is the reflection of the ellipse for the \( D^{*'} < 0 \) with respect to the axis \( D_r = 1 \). The ellipses converge to the point \( M(\beta_r = 0.556, D_r = 1) \) when \( D^{*'} = \pm 0.027 \) and \( \alpha^* = 0 \). We found the ellipses shown in Fig. 2.4 by solving Eq. (2.52) for constant values of \( I \).

The variables \( \beta_r, D_r, \alpha^* \), which meet the Eq. (2.58), constitute a closed surface for given values of \( D^{*'} \) and \( \epsilon_r \). According to Eq. (2.52), the closed surfaces for the \( D^{*'} = \{-0.01, 0, 0.01\} \) and \( \epsilon_r = \{1.41, 2, 3, 4\} \) were computed. They are shown in Fig. 2.5. Figure 2.5a presents the closed surfaces which are cut off by the planes \( D_r = 0 \) and \( \alpha^* = 1.75 \). The other figures shown are cut off by the planes \( D_r = 0 \). As one can see from Figs. 2.5a - 2.5b, the surfaces without dispersion derivative are symmetrical with respect to the planes.
$D_r = 1$ and $\alpha^* = 0$ but in the presence of dispersion derivative $D'' \neq 0$ the symmetry is broken.

### 2.6.2 Zero dispersion and its derivation at the entrance of the bending magnet

In the case $D_0 = 0$, $D_0' = 0$ at the entrance of the bending magnet, the integral in Eq. (2.51) takes a minimum value when the $\beta_0$ and $\alpha_0$ at the entrance of the magnet are

$$\beta_{0}^{\text{opt}} = 2L\sqrt{\frac{3}{5}}, \quad \alpha_{0}^{\text{opt}} = \sqrt{\frac{15}{\sqrt{15}}} \approx 3.873$$  \hspace{1cm} (2.59)

It yields the emittance of

$$\gamma_{x0}^{\text{opt}} = \frac{C_q\gamma^3}{J_x} \frac{\theta^3}{4\sqrt{15}}$$  \hspace{1cm} (2.60)

As one can see, the emittance given by Eq. (2.60) is three times bigger than the theoretical minimum emittance $\gamma_{x0m}$ given by Eq. (2.60). Figure 2.6 shows how the emittance changes with variations of $\alpha_0$ and $\beta_0$ away from their optimum values.

![Figure 2.6](image)

**Figure 2.6:** a) Ellipses of constant detuning factors $\epsilon_{x0}/\epsilon_{x0m}$ of emittance as a function of the deviation from $\beta_0^{\text{opt}}$ and $\alpha_0^{\text{opt}}$; b) detuning factors versus beta function for a given $\alpha_0$.

The detuning factor $\epsilon_r$ with respect to the theoretical minimum emittance is given:

$$\epsilon_r = \frac{\epsilon_{x0}}{\epsilon_{x0m}} = 3 \cdot \frac{1 + \left(\sqrt{15} + \Delta \alpha_0\right)^2}{2\beta_r} + 24\beta_r - 3\left(15 + \Delta \alpha_0\sqrt{15}\right)$$  \hspace{1cm} (2.61)

where $\Delta \alpha$ is deviation from optimal value $\alpha_0^{\text{opt}} = \sqrt{\frac{15}{\sqrt{15}}}$ and $\beta_r \equiv \beta_0/\beta_0^{\text{opt}}$.

It is useful to note, that the minimum value of emittance is achieved when the minimum ($\alpha_f = 0$) of the horizontal beta function within the dipole occurs at a distance $s_f = L \cdot 3/8$ from the beginning of the magnet and the value of the minimum betatron function at $s_f$ is:

$$\beta_f = L\sqrt{\frac{3}{8\sqrt{5}}}$$  \hspace{1cm} (2.62)
2.7 High brilliance lattice types

A small emittance can be achieved with different magnet lattices. The basic structure of lattice for the modern light source consists of an achromat ending in bending magnets on either side and two adjacent straight sections to provide dispersion-free sections for the installation of insertion devices (wigglers) that allows to avoid emittance blow-up by wigglers. The main difference between lattice structure of light sources and damping rings is the periodicity (number of identical lattice cells). The lattice structure of light sources have to provide many dispersion-free sections with wigglers in order to have a big number of synchrotron radiation outlet channels from wigglers. The damping ring structure should not provide this feature and usually it has two long dispersion-free straight sections with wigglers connected by the arcs. Such scheme of design is called ”racetrack”. It allows to achieve a better minimization of emittance in the arcs.

There are several types of low emittance lattices generally used in modern light sources. They are briefly described below.

2.7.1 Double Focusing Achromat (DFA)

Double Focusing Achromat (DFA) are commonly known as Chasman-Green [21] and Expanded Chasman-Green lattice. The double focusing achromat lattice has been used for the NSLS rings in Brookhaven [22]. An expanded Chasman-Green structure is the basis of the conceptual designs of several synchrotron radiation sources: ESRF [23], APS [24], ELETTRA [25], SUPERACO [26] and SOLEIL [27].

The double focusing achromat lattice or basic Chasman-Green represents a compact structure used in low emittance storage rings. The basic scheme uses two dipole magnets surrounding a focusing quadrupole. The strength of the quadrupole is adjusted so that the dispersion generated by the first dipole is cancelled by passing through the second dipole. In this form, the structure is not flexible since the quadrupole does not provide focusing in both planes. Therefore, in the dispersion region, defocusing quadrupoles must be added upstream and downstream of the focusing quadrupole. For example, the ESRF as well as SUPERACO have four quadrupoles in the dispersion region. The DFA structure of SUPERACO is shown in Fig. 2.7. This optics represents the so-called expanded Chasman-Green achromat. A few focusing and defocusing quadrupoles are located in the dispersion free straight section (insertion sections) where a wiggler magnet is inserted. The minimum emittance for the DFA lattice is given by Eq. (2.60) if the horizontal betatron function satisfy to the requirements of Eq. (2.59) or Eq. (2.62). The minimum emittance for DFA is three times larger than the theoretical minimum emittance given by Eq. (2.60).
2.7.2 Triplet Achromat Lattice (TAL)

The triplet achromat lattice was used in the storage ring ACO at Orsay [28]. TAL lattice can be made very compact since there are no quadrupoles in the dispersion free straight sections. The minimum emittance of the TAL is given as

$$\gamma \epsilon_{x0}^{TAL} = \frac{C_q \gamma^3}{J_x} \theta^2 \frac{2}{3} \left( \frac{\beta_x}{L_{opt}} \right)$$

where \(L\) is the length of bending magnets. The optimum value of the horizontal betatron function in the middle of the dispersion free straight section of length \(2L_i\) is

$$\left( \frac{\beta_x}{L} \right)_{opt}^2 = \frac{3}{4} \left[ \frac{1}{5} + \frac{L_i}{L} + \frac{4}{3} \left( \frac{L_i}{L} \right)^2 \right].$$

At the extreme case when \(L_i \rightarrow 0\), the minimum emittance of the TAL is 12 times larger than the theoretical minimum emittance. The main disadvantage of the lattice is that the emittance depends on the value of the \(\beta_x\) in the insertion region.

2.7.3 Triplet Bend Achromat (TBA)

Triple bend structures are utilized at the following synchrotron radiation sources; ALADDIN [29], BESSY [30], ALS [31] in Berkeley, SRRC [32] in Taiwan, and PLS [33]. They were also proposed for the DIAMOND project [34].

The triple bend achromat lattice is the logical extension of the DFA. Insertion of a third bending magnet within the DFA (for example, between defocusing quadrupoles of the achromat in Fig. 2.7) allows one to reduce the minimum emittance and to have extra...
flexibility. One part of the emittance of a TBA, which is produced in the two outer magnets, is equal to the emittance of the DFA structure. The second part arises in the inner magnet. Assuming equal bending angle for all three magnets, the minimum emittance is obtained as [35]

\[ \gamma^TBA_{x0} = \frac{7}{36\sqrt{15}} \frac{C_q\gamma^3 \theta^3}{J_x} \]  

if in the middle of the inner magnet the lattice functions are chosen as

\[ \beta_x = \frac{L_{inn}}{\sqrt{15}} \quad D_x = \frac{L_{inn}}{6\rho} \]

\[ \alpha_x = 0 \quad D'_x = 0 \]

and lattice functions in the outer bending magnets satisfy the requirements for the DFA given by Eq. (2.59) or Eq. (2.62). However, lower emittance value can be obtained if the bending angle of the inner magnet is larger by factor of 1.5 than the bending angle of the outer magnets.

2.7.4 Theoretical minimum emittance lattice (TME)

The TME lattice is based on the optical symmetry of the horizontal beta and dispersion functions with respect to the center of the bending magnets that was discussed in Sec. 2.6.1. The TME lattice was proposed for most of the damping rings developed for the future linear collider projects, for example, TESLA damping ring [36], NLC damping ring [37, 38], GLC damping ring [39] and afterwards some possible variants of damping ring for the International Linear Collider ILC [40].

A TME cell is composed of one bending magnet and several (typically 3 - 4) quadrupole magnets. For a TME cell with small bending angle \( \theta \ll 1 \) and optical symmetry with respect to the middle of the bending magnet, the equilibrium emittance is given by Eq. (2.56).

The emittance detuning factor depends on the phase advance per the TME cell. If the conditions \( D'_x = 0 \) and \( D^*_x/D^*_m = D^*_{mx} \) are true at the bending center (see Eq. 2.56) then the detuning factor is uniquely given by the horizontal phase advance per the TME cell as [41]

\[ \tan\left(\frac{\mu_x}{2}\right) = \frac{\epsilon_r\sqrt{3}}{\sqrt{\epsilon^2_r - 1 - \sqrt{5}}} \]  

Thus, a phase advance per TME cell of \( \mu_x \approx 284^\circ \) produces the smallest emittance of \( \epsilon_r = 1 \). The value of detuning factor becomes infinite, \( \epsilon_r \to \infty \), when \( \mu_x \) approaches 120°. The relationship between phase advance and detuning factor is summarized in Fig. 2.8. Accordingly, the choice of the emittance detuning factor is simply defined by choosing the horizontal phase advance per TME cell.
2.8 Choices of lattice type for the damping ring

To attain the very low emittances needed for the CLIC damping ring, the lattice should be efficient and have a small $I_5$ integral for a given bending magnet strength. As one can see from Sec. 2.7, many possible lattice choices have been developed for the low emittance synchrotron radiation sources. The DFA and TBA lattices were originally designed to have dispersion-free straight sections after every pair or every triplet of bending magnets, respectively.

However, the needs of the damping rings are different from that of the synchrotron radiation sources. In particular, one does not need many dispersion-free straight sections for insertion devices. In the damping rings, we need two dispersion-free regions for injection/extraction and damping wigglers. The DFA, TAL and TBA lattices are not really optimized to create compact and efficient $180^\circ$ arcs consisting of many cells. Moreover, the minimum achievable emittance of these lattice is a few times bigger than that for the TME:

\[
\frac{\epsilon_{x0}^{TAL}}{\epsilon_{x0}^{TME}} = 12 \quad \frac{\epsilon_{x0}^{DFA}}{\epsilon_{x0}^{TME}} = 3 \quad \frac{\epsilon_{x0}^{TBA}}{\epsilon_{x0}^{TME}} = \frac{7}{3}
\]

For this reason, we will consider only the TME lattice for the CLIC damping rings.

Another consideration is the choice of the horizontal damping partition number $J_x$. By using a combined function bend with a defocusing gradient, it is possible to increase $J_x$ (see Eq. 2.57) reducing the equilibrium horizontal emittance and the horizontal damping time $\tau_x$. However, combined function magnets can be more difficult to align and have tighter field tolerances. As it was mentioned in Sec. 2.6.1, if the radiation damping is dominated by the wigglers, the relative gain from changing $J_x$ is small. For these reasons, we will not consider a combined function magnet, although the initial variant of the lattice for the CLIC damping ring documented in Ref. [42], was based on the TME structure with combined function bending magnets.
The choice of horizontal phase advance per TME cell is very important. Increasing of \( \mu_x \) on the one hand yields a lower emittance but on the other hand it decreases the average value of the lattice functions, as a consequence, making it difficult to compensate large natural chromaticity. With the small optical functions, the required strength of sextupoles becomes very strong. Strong sextupoles limit the dynamic aperture of the machine (the maximum amplitude of the stable betatron oscillations). One can suppose that a long TME cell may provide a high horizontal phase advance and relatively big lattice functions. This is true if we do not take into account the effect of *intrabeam scattering* that will be discussed in Chapter 3. As it will be seen, the Intrabeam scattering has a strong impact on the equilibrium emittance. In order to minimize this impact, the damping times must be small. Thus, at a fixed number of bending magnets, the damping ring circumference \( C \) and the length of bending magnets \( L \) have to be as small as possible since the damping times are directly proportional to the revolution time \( T_0 \) and directly proportional to \( \rho = L/\theta \) (see Eq. 2.28 and Eq. 2.30). In our opinion the TME lattice is the best choice to construct very short arcs producing low emittance.

The chromaticity correction and non-linear optimization of the damping ring will be studied in the Chapter 5 "Non-linear optimization of the CLIC damping ring lattice". However, at the stage of linear optics design, we have provided some flexibility which enables us to perform a nonlinear optimization, that means the possibility to arrange second order sextupolar achromat and sextupole families with \(-I\) separation between sextupoles. We will choose the horizontal and vertical betatron phase advance per TME in the range of \( 180^\circ - 270^\circ \) in order to provide both low emittance and the possibility of arranging second order sextupolar achromats.

The four-quad TME cell produces smaller beta function peaks and thus leads to a smaller peak beam size. In addition, "four-quad" variant of the TME lattice provides good positions between defocusing quads for the sextupoles assigned to correct the vertical natural chromaticity because in this place the horizontal and vertical beta functions are sufficiently different which reduces the strength of the sextupoles. We will consider a four-quad TME cell with focusing quad (FQ) located near both ends of the bending magnet (B) and with a pair of defocusing quads (DQ) located between the focusing quads, i.e. the structure of one TME cell providing horizontal phase advance \( \mu_x > 180^\circ \) and vertical phase advance \( \mu_y < 180^\circ \) is \( s_1-[DQ]-s_2-[FQ]-s_3-[B]-s_3-[FQ]-s_2-[DQ]-s_1 \) where \( s_1, s_2, s_3 \) are drift spaces. The pair of defocusing quads have equal strength. The strength of the focusing quads are equal too. The derivatives of the lattice functions take zero value between the defocusing quadrupoles. If the polarity of the quadrupoles is changed, we get TME lattice where \( \mu_x < 180^\circ \). This variant is not considered because of large emittance. In summary, we have chosen

- The compact four-quadrupole TME cell with short bending magnets and \( \mu_x > 180^\circ \)

2.9 Choices of the damping ring energy

As CLIC will operate with polarized beams, the damping ring must maintain a high spin polarization. Therefore, the ring energy should be chosen so that the spin tune is a half integer to stay away from the strong integer spin resonances. This constrains the ring energy to

\[
a\gamma = n + \frac{1}{2}
\]
Here, \( a = 1.16 \times 10^{-3} \) is the anomalous magnetic moment of the electron (or positron) and \( n \) is the integer numbers. This limits the possible energy to 1.54 GeV \((n = 3)\), 1.98 GeV \((n = 4)\) that is the design energy of NLC and GLC damping rings, 2.42 GeV \((n = 5)\), 2.86 GeV \((n = 6)\), 3.3 GeV \((n = 7)\) and so on. For example, the design energy for the TESLA and ILC damping rings was chosen to 5 GeV that is very close to \((n = 11)\).

Which energy to choose? Let us scale the damping ring parameters and beam parameters with respect to the energy. In other words, we would like to estimate the dependence of damping ring parameters on beam energy for the fixed normalized target emittance \( \gamma \epsilon_{x0} = 450 \) nm. The number \( N_T \) of TME cells, required to get the normalized target emittance of 450 nm, is given by solving Eq. (2.56) for \( \theta \)

\[
N_T = \frac{2\pi}{\theta} = 2\pi \gamma \left( \frac{\epsilon_r}{\sqrt{15}} \right)^{1/3} \left[ \frac{12\gamma \epsilon_{x0} J_x}{C_q} \right]^{-1/3} = 0.016576 \gamma \epsilon_r^{1/3} \tag{2.67}
\]

The bending angle is inversely proportional to the energy; \( \theta \propto 1/\gamma \). We assume that all bending magnets are identical. To maintain high damping, we consider a short bending magnets with high magnetic field which is related with energy and length of bending magnets as \( B_a = [\theta(B\rho)/L] \). Let us keep constant length \( L \) of the bending magnets for any energy. In this case, the bending radius changes with energy as \( \rho = L/\theta \propto \gamma \), if the magnetic field \( B_a \) is constant. According to Eq. (2.49), the dispersion invariant depends on energy as \( \langle H_x \rangle \propto 1/\gamma^2 \). If the emittance detuning factor is fixed for any energy, the dispersion and beta function in the middle of the magnet are scaled as \( D^* \sim L\theta/24 \propto 1/\gamma \) and \( \beta^* \propto L \). Inserting these dispersion and beta functions in the Eq. (2.52), it is easy to see that \( \langle H_x \rangle \) depends on energy as \( 1/\gamma^2 \). Assuming that the length of TME cells does not change, the energy loss per one turn, the circumference of the ring, transverse damping time, and the momentum compaction must scale as

\[
U_0 \propto \frac{\gamma^4}{\rho} \propto \gamma^3 \quad C \propto N_T \propto \gamma \quad \tau \propto \frac{\gamma C}{U_0} \propto \frac{1}{\gamma} \quad \alpha_r \propto \frac{1}{\gamma^2} \quad \text{at} \quad L \equiv \text{const}, \quad B_a \equiv \text{const}
\]

Using Eq. (2.45–2.46), one can see that

\[
\sigma_\delta \propto \sqrt{\gamma} \quad \text{and} \quad \sigma_{x0} \propto 1/\gamma
\]

The dispersion invariant is changed due to dispersion because \( \beta_x \propto L \). However, for the length of bending magnet \( \sim 0.5 \div 1 \) m at the length of TME cell \( \sim 2 \div 4 \) m and \( \mu_x > 180^\circ \) the average value of \( \beta_x \) is usually \( \sim 10 \div 100 \) times bigger than the average value of \( D_x \). The horizontal dispersion in modern low emittance damping ring is usually much less than one meter.

The average value of \( D_x \) is reduced with energy which causes a problem with dynamic aperture due to the need of very strong sextupoles for the chromatic correction. From the point of view of intra-beam scattering, the IBS grows times are decreased with energy, if the parameters of the ring are changed with energy. Because the emittance growth due to intrabeam scattering depends on the scattering growth time compared to the damping time, the intrabeam scattering actually becomes worse as the design energy is increased. In the energy range \( 2 \div 2.5 \) GeV, our preliminary estimation shown that due to the IBS the value of final normalized equilibrium emittance for the damping ring consisting of 2 m of long TME cells with \( L=0.5 \) m and \( \epsilon_r = 1.5 \) is about two times larger than the value of target equilibrium emittance of 450 nm defined only by quantum excitation and radiation damping.
In order to compensate an emittance increase due to IBS, a wiggler magnets are needed to increase radiation damping. At the presence of wiggler, the damping times are decreased as \( \tau/(1 + F_w) \) where \( F_w \sim L_w B_w^2 / \gamma B_a \). The parameters \( B_w \) and \( L_w \) are wiggler field and the total length of the wigglers. The number of wigglers is proportional to the energy as \( \propto 1/\gamma \).

The damping ring design for high energy will need a large number of wigglers.

The cost of the rings will tend to increase with the number of cells, while the cost of the RF systems will increase with the power required; both of these costs will increase with higher energy. In addition, the momentum compaction decreases with the square of the ring energy while rough scaling for the longitudinal microwave threshold scales as \( \gamma \alpha_p \). This suggests that longitudinal stability may be more difficult at higher energy.

The minimum number of TME cells given by Eq. (2.67) at \( \epsilon_r = 1 \) for the 1.98 GeV, 2.42 GeV and 2.86 GeV are 66, 80 and 94 units, respectively. Taking into account above-stated reasoning, we have chosen an nominal energy for the CLIC damping ring of 2.42 GeV. This is the energy that appears to yield reasonable designs for the bending and quadrupole magnets and the wigglers.

In the next section we describe an effect of intra-beam scattering because it becomes very strong for the ultra-low emittance machines.
Chapter 3
Intrabeam scattering

3.1 Introduction

Intrabeam scattering (IBS) involves multiple small-angle Coulomb scatterings between the charged particles of accelerator beams. This phenomenon leads to the growth in beam emittances, which places severe limitations on luminosity lifetimes in hadron and heavy ion colliders and the ability to achieve ultrasmall beam emittances in intense electron storage rings.

In electron (positron) storage rings, the effect of intrabeam scattering (IBS) leads to an increase in the six dimensional emittance of the bunch. Roughly speaking, the increase of the six dimensional emittance due to IBS depends on transverse beam size, rms bunch length, and relative energy spread which are defined by radiation damping and quantum excitation and RF. Furthermore, IBS depends on the bunch charge, beam energy and dispersion functions along the ring. IBS is a very important effect in electron (positron) low emittance damping rings [43] and synchrotron light sources, as well as in hadronic [44] and heavy ion [45] circular machines.

The IBS is different from the Touschek-Effect [46] which is also caused by Coulomb scattering. The Touschek-Effect, however, is a single scattering effect where the energy transfer from the horizontal to the longitudinal direction leads to the loss of the colliding particles. The IBS is essentially a diffusion process in all three dimensions. Collisions between particles in a bunch may lead to a small enough transfer of momentum, that the particles involved are not lost from the beam. In this case, there is an increase in the energy spread of the bunch, which couples back through the dispersion into the transverse planes.

A change in the momentum deviation of a particle in a dispersive region of the ring results in a change of its betatron oscillation amplitude. The growth rate of the emittance due to either IBS and quantum excitation then follows from a consideration of the statistics of the transverse excitation. An increase of the transverse beam emittance through quantum excitation occurs only when synchrotron radiation is emitted at a place with nonzero dispersion. The emittance growth due to IBS is similar, but in contrast to synchrotron radiation it also arises outside of the bending magnets.

The evolution of electron (positron) beam emittances in the CLIC damping ring is defined
mainly by the interplay of radiation damping, quantum excitation, and intra-beam scattering (IBS). The horizontal emittance \( \epsilon_x \), vertical emittance \( \epsilon_y \) and rms relative energy spread \( \sigma_p \) evolve with time according to a set of three coupled differential equations [47]:

\[
\frac{d\epsilon_x}{dt} = -\frac{2}{\tau_x} (\epsilon_x - \epsilon_{x0}) + \frac{2\epsilon_x}{T_x(\epsilon_x, \epsilon_y, \sigma_p)}
\]
\[
\frac{d\epsilon_y}{dt} = -\frac{2}{\tau_y} (\epsilon_y - \epsilon_{y0}) + \frac{2\epsilon_y}{T_y(\epsilon_x, \epsilon_y, \sigma_p)}
\]
\[
\frac{d\sigma_p}{dt} = -\frac{1}{\tau_p} (\sigma_p - \sigma_{p0}) + \frac{\sigma_p}{T_p(\epsilon_x, \epsilon_y, \sigma_p)}
\]

where \( \epsilon_{x0}, \epsilon_{y0} \) and \( \sigma_{p0} \) are the horizontal and vertical zero-current equilibrium emittances and rms relative energy spread, respectively, which are determined by radiation damping and quantum excitation in the absence of IBS. \( \tau_x, \tau_y, \tau_p \) are the radiation damping times of the betatron \((x, y)\) and synchrotron \((p)\) oscillations, respectively. \( T_x, T_y, T_p \) are the horizontal, vertical and longitudinal IBS growth times. The differential equations (3.1) are coupled through the IBS growth times \( T_\mu(\epsilon_x, \epsilon_y, \sigma_p), \mu \in \{x, y, p\} \), which are non-linear functions of \( \epsilon_x, \epsilon_y \) and \( \sigma_p \). The equilibrium emittances follow from the solution of the following equation

\[
\frac{d\epsilon_x}{dt} = \frac{d\epsilon_y}{dt} = \frac{d\sigma_p}{dt} = 0
\]

The first rather thorough treatment of IBS for accelerators was developed by Piwinski [48]. This result was extended by Martini [49], giving the so-called the standard Piwinski (P) method [50]. Another formalism was detailed by Bjorken and Mtingwa (B-M) [51]. Both approaches solve the local, two-particle Coulomb scattering problem for (six-dimensional) Gaussian beams (at weak betatron coupling or for uncoupled beam), though a more generalized formulation, which includes linear coupling and can also be applied to low emittance machines, is given by Piwinski in Ref [52]. The B-M result is considered to be more general, both P and B-M approaches give different values of growth times at very low emittance. The B-M method is more often used in modern optics software codes, for example, such as SAD [53] and MAD. However, CPU time to compute the IBS growth times by both P and B-M methods can be quite long since at each iteration a numerical integration at every lattice element is needed. Thus, over the years, many authors have attempted to derive high energy approximations to the full theory in order to simplify the IBS calculations. For instance, approximate solutions were described by Parzen [54], Le Duff [55], Raubenheimer [56], and Wei [57].

K. Bane [58] has suggested to modify the standard form of Piwinski theory (in the following, we will call his modification the ”modified Piwinski theory”). He has also derived a high energy approximation [59] to the B-M theory and has demonstrated its equivalence to the high energy limit of the modified Piwinski theory.

### 3.2 The general Bjorken and Mtingwa solution

For bunched beams, the growth times according to Bjorken-Mtingwa (including vertical dispersion) are

\[
\frac{1}{T_i} = 4\pi A (\log) \left\{ \int_0^\infty d\lambda \frac{\lambda^{1/2}}{[\det(L + \lambda I)]^{1/2}} \left\{ \text{Tr} L^{(i)} \text{Tr} \left( \frac{1}{L + \lambda I} \right) - 3 \text{Tr} L^{(i)} \left( \frac{1}{L + \lambda I} \right) \right\} \right\} ,
\]

30
where $i$ represents $p$, $x$, or $y$ and $\langle \cdots \rangle$ indicates that the integral is to be averaged around the accelerator lattice. The 6-dimensional invariant phase space volume of a bunched beam are

$$A = \frac{cN\lambda^2}{64\pi^2\beta^3 \gamma^4 \epsilon_x \epsilon_y \sigma_x \sigma_y}.$$  \hfill (3.4)

Here, $r_0$ the classical particle radius (2.82 \times 10^{-15} \text{ m for electron or positron beam, 1.53 \times 10^{-18} \text{ m for proton beam}), c$ the speed of light, $N$ the number of particles per bunch (bunch population), $\beta$ is the particle speed divided by $c$, $\gamma$ the Lorentz energy factor, and $\sigma$ the rms bunch length; det and Tr signify, respectively, the determinant and the trace of a matrix, and $I$ is the unit matrix. The $(\log) \equiv \ln\left(\frac{r_{\text{max}}}{r_{\text{min}}}\right)$ is the Coulomb logarithm that is the ratio of the maximum $r_{\text{max}}$ to the minimum $r_{\text{min}}$ impact parameter in the collision of two electrons in the bunch. For typical flat beam, the $r_{\text{max}}$ is taken to be equal to the vertical beams size $\sigma_y$, while $r_{\text{min}}$ is taken to be equal $r_0\beta_x/(\gamma^2\epsilon_x)$. In this case, the Coulomb logarithm may be estimated as

$$(\log) = f_{\text{CL}} \ln\left(\frac{\gamma^2\epsilon_x\sqrt{\beta_y\epsilon_y}}{r_0\beta_x}\right).$$  \hfill (3.5)

For Gaussian bunches, the factor $f_{\text{CL}} = 1$. However, IBS populates the tails of the bunch distribution, and this leads to a reduction in the growth rates of the core emittances; this may be represented by a reduction in the factor $f_{\text{CL}}$ to a value as low as 0.5 [60].

The auxiliary matrices in Eq. (3.3) are defined as

$$L = L^{(p)} + L^{(x)} + L^{(y)}$$  \hfill (3.6)

$$L^{(p)} = \frac{\gamma^2}{\sigma_p^2} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \hfill (3.7)$$

$$L^{(x)} = \frac{\beta_x}{\epsilon_x} \begin{pmatrix} 1 & -\gamma\phi_x & 0 \\ -\gamma\phi_x & \gamma^2 H_x/\beta_x & 0 \\ 0 & 0 & 0 \end{pmatrix} \hfill (3.8)$$

$$L^{(y)} = \frac{\beta_x}{\epsilon_x} \begin{pmatrix} 0 & 0 & 0 \\ 0 & \gamma^2 H_y/\beta_y & -\gamma\phi_y \\ 0 & -\gamma\phi_y & 1 \end{pmatrix} \hfill (3.9)$$

where in the above expressions, the function $\phi_{x,y}$ are given as

$$\phi_{x,y} = D'_{x,y} - \frac{\beta'_{x,y} D_{x,y}}{2\beta_x}.$$  \hfill (3.10)

### 3.3 Bane’s high energy approximation

With a change of the integration variable $\lambda$ in Eq. (3.3) to $\lambda' = \lambda \sigma_H^2/\gamma^2$, Bane obtains the following high energy approximations [59]:

\[ \text{31} \]
\[
\frac{1}{T_p} \approx \frac{r_0^2 c N (\log)}{16 \gamma^3 \epsilon_x^{3/4} \epsilon_y^{3/4} \sigma_s \sigma_p^3} \left( \sigma_H g \left( \frac{a}{b} \right) (\beta_x \beta_y)^{-1/4} \right)
\] (3.11)

and

\[
\frac{1}{T_{x,y}} \approx \frac{\sigma_p^2 \mathcal{H}_{x,y}}{\epsilon_{x,y}} \frac{1}{T_p}
\] (3.12)

where

\[
\frac{1}{\sigma_H^2} = \frac{1}{\sigma_p^2} + \frac{\mathcal{H}_x}{\epsilon_x} + \frac{\mathcal{H}_y}{\epsilon_y}
\] (3.13)

\[
a = \frac{\sigma_H}{\gamma} \sqrt{\frac{\beta_x}{\epsilon_x}}
\] (3.14)

\[
b = \frac{\sigma_H}{\gamma} \sqrt{\frac{\beta_y}{\epsilon_y}}
\] (3.15)

The function \( g \) in Eq. (3.11) is given by the elliptic integral

\[
g(\alpha) = \frac{2}{\pi} \int_0^\infty \frac{du}{\sqrt{1 + u^2} \sqrt{\alpha^2 + u^2}}
\] (3.16)

where \( \alpha = a/b \). A requirement of the high energy approximation is that \( a, b \ll 1 \). If the momentum of particles in the longitudinal plane is much less than in the transverse planes, this requirement is satisfied. A second assumption is that \( \phi_{x,y} \sigma_H \sqrt{\beta_{x,y}/\epsilon_{x,y}} < 1 \) in order to drop off-diagonal terms in Eq. (3.8–3.9).

For flat beams \( a/b \) is less than 1. The function \( g(\alpha) \), related to the integral (3.16), can be well approximated by

\[
g(\alpha) = \alpha^{0.021 - 0.044 \ln \alpha} \quad \text{for the limit } [0.01 < \alpha < 1]
\] (3.17)

to obtain \( g \) for \( \alpha > 1 \), note that \( g(\alpha) = g(1/\alpha) \). The fit (3.17) has a maximum error of 1.5% over \([0.02 < \alpha < 1]\). We may assume that the vertical zero-current equilibrium emittance \( \epsilon_{y0} \) in Eqs. (3.11–3.12) is determined mainly by the spurious vertical dispersion.

Raubenheimer’s approximation formula [56] is similar, though less accurate, than Eq. (3.11). In Raubenheimer’s approximation, the expression \( g(a/b) \sigma_H / \sigma_p \) in Eq. (3.11) is replaced by the factor 1/2.

### 3.4 The standard Piwinski solution

The standard Piwinski theory of intrabeam scattering is summarized nicely in Ref. [50]. The relative energy spread and transverse emittance growth times are given by
\[
\frac{1}{T_p} = A \left\langle \frac{\sigma_p^2}{\sigma_p^2} f(\tilde{a}, \tilde{b}, \tilde{q}) \right\rangle 
\]  
(3.18)

\[
\frac{1}{T_x} = A \left\langle f \left( \frac{1}{\tilde{a}}, \frac{\tilde{b}}{\tilde{a}}, \frac{\tilde{q}}{\tilde{a}} \right) + \frac{D_x^2 \sigma_h^2}{\beta_x \epsilon_x} f(\tilde{a}, \tilde{b}, \tilde{q}) \right\rangle 
\]  
(3.19)

\[
\frac{1}{T_y} = A \left\langle f \left( \frac{1}{\tilde{b}}, \frac{\tilde{a}}{\tilde{b}}, \frac{\tilde{q}}{\tilde{b}} \right) + \frac{D_y^2 \sigma_h^2}{\beta_y \epsilon_y} f(\tilde{a}, \tilde{b}, \tilde{q}) \right\rangle 
\]  
(3.20)

where \( A \) is defined the same as in Eq. (3.4) and

\[
\frac{1}{\sigma_h^2} = \frac{1}{\sigma_p^2} + \frac{D_x^2}{\beta_x \epsilon_x} + \frac{D_y^2}{\beta_y \epsilon_y} 
\]  
(3.21)

\[
\tilde{a} = \frac{\sigma_h}{\gamma} \sqrt{\frac{\beta_x}{\epsilon_x}} 
\]  
(3.22)

\[
\tilde{b} = \frac{\sigma_h}{\gamma} \sqrt{\frac{\beta_y}{\epsilon_y}} 
\]  
(3.23)

\[
\tilde{q} = \sigma_h \bar{\beta} \sqrt{\frac{2d}{r_0}} 
\]  
(3.24)

The maximum impact parameter \( d \) is usually taken to be the vertical beam size. The original Piwinski scattering function \( f \) is defined in Ref. [48]. A single integral representation of \( f \), that has a more simple form than the original one, was given some time ago by Evans and Zotter [61] as

\[
f(\tilde{a}, \tilde{b}, \tilde{q}) = 8\pi \int_0^1 du \left\{ \frac{(1 - 3u^2)}{PQ} \right\} \left\{ 2 \ln \left[ \frac{q}{2} \left( \frac{1}{P} + \frac{1}{Q} \right) \right] - 0.577 \cdots \right\} 
\]  
(3.25)

with

\[
P^2 = \tilde{a}^2 + (1 - \tilde{a}^2)u^2 
\]  
(3.26)

\[
Q^2 = \tilde{b}^2 + (1 - \tilde{b}^2)u^2 
\]  
(3.27)

where the function \( f \) satisfies the following relations:

\[
f(\tilde{a}, \tilde{b}, \tilde{q}) = f(\tilde{b}, \tilde{a}, \tilde{q}) 
\]  
(3.28)

\[
f(\tilde{a}, \tilde{b}, \tilde{q}) + \frac{1}{\tilde{a}^2} f \left( \frac{1}{\tilde{a}}, \frac{\tilde{b}}{\tilde{a}}, \frac{\tilde{q}}{\tilde{a}} \right) + \frac{1}{\tilde{b}^2} f \left( \frac{1}{\tilde{b}}, \frac{\tilde{a}}{\tilde{b}}, \frac{\tilde{q}}{\tilde{b}} \right) = 0. 
\]  
(3.29)
3.5 The modified Piwinski formulation

The Piwinski’s solution (3.18–3.27) depends on $D^2/\beta$, and not on dispersion invariant $\mathcal{H}$ as the general B-M solution (3.3–3.10). K. Bane [59] suggested to replace $D^2/\beta$ in Eqs. (3.18–3.21) by $H$:

$$\frac{D^2_{x,y}}{\beta_{x,y}} \rightarrow \mathcal{H}_{x,y} = \frac{1}{\beta_{x,y}} \left[ D^2_{x,y} + \left( \beta_{x,y} D'_{x,y} - \frac{1}{2} \beta'_{x,y} D_{x,y} \right)^2 \right]$$

(3.30)

which means $\sigma_{h, \tilde{a}, \tilde{b}}$ in Eqs. (3.21–3.23) become $\sigma_{H, a, b}$ from Eqs. (3.13–3.15):

$$\frac{1}{\sigma^2_p} = \frac{1}{\sigma^2_p} + \frac{D^2_x}{\beta_x \epsilon_x} + \frac{D^2_y}{\beta_y \epsilon_y} \rightarrow \frac{1}{\sigma^2_H} = \frac{1}{\sigma^2_p} + \frac{\mathcal{H}_x}{\epsilon_x} + \frac{\mathcal{H}_y}{\epsilon_y}$$

(3.31)

The Piwinski formulation described in Sec. 3.4 with the replacements (3.30–3.31) is called as the modified Piwinski formulation.

3.6 Equilibrium emittances due to IBS

Without IBS, the evolution of the three emittances after injection into the damping ring to subsequent extraction is given by

$$\gamma \epsilon_{\text{ext}} = e^{-\frac{2}{\tau}} \gamma \epsilon_{\text{inj}} + \left(1 - e^{-\frac{2}{\tau}}\right) \gamma \epsilon_0$$

(3.32)

where $\gamma \epsilon_{\text{inj}}, \gamma \epsilon_{\text{ext}}, \gamma \epsilon_0$ are, respectively, the injected, extracted and equilibrium normalized emittances; $t$ is the time after injection, and $\tau$ is the damping time.

Taking into account IBS, the steady-state beam emittances and relative energy spread obtained by solving Eqs. (3.1–3.2) in the presence of spurious vertical dispersion and in the limit of weak betatron coupling satisfy the following conditions [62]

$$\epsilon_x = \frac{\epsilon_0}{1 - \tau_x/T_x}, \quad \epsilon_y = \epsilon_0 \left[ \frac{1 - r_x}{1 - \tau_y/T_y} + \frac{r_x}{1 - \tau_x/T_x} \right], \quad \sigma^2_p = \frac{\sigma^2_{p,0}}{1 - \tau_p/T_p}$$

(3.33)

where

$$r_x = \frac{\epsilon_{y,0,\beta}}{\epsilon_{y,0}} = \frac{\epsilon_{y,0,\beta}}{\epsilon_{y,0,d} + \epsilon_{y,0,\beta}} = \frac{\kappa \epsilon_{x,0}}{\epsilon_{y,0,d} + \kappa \epsilon_{x,0}}$$

(3.34)

In Eq. (3.34), we ignore a contribution from the vertical opening angle of the radiation, since for high energy beams, even for CLIC, it is always small compared to the contributions from other sources. As one can see from the above sections, all three IBS rise times are coupled through the $\epsilon_x, \epsilon_y$ and $\sigma_p$. Note that the rms bunch length $\sigma_s$ is directly proportional to the momentum spread $\sigma_p$. Generally this is taken to be the nominal (zero current) relationship given by Eq. (2.41).

If there is only $x - y$ betatron coupling ($\langle \mathcal{H}_y \rangle = 0 \Rightarrow \epsilon_{y,0,d} = 0$) then $r_x = 1$. In this case the steady-state vertical emittance is equal to $\epsilon_y = \kappa \epsilon_{x,0}/(1 - \tau_x/T_x)$. In the presence of vertical dispersion only ($\kappa = 0$), the parameter $r_x$ becomes equal to 0 and Eq. (3.33) for the vertical emittance reduces to the expression $\epsilon_y = \epsilon_{y,0,d}/(1 - \tau_y/T_y)$. 

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Piwinski suggested to iterate numerically Eqs. (3.33) until a self-consistent solution is found. However, it might cause some problems, namely, nonsensical negative values of the emittance or of $\sigma^2_p$ can be obtained with this procedure. From our point of view, it is better to solve numerically the three coupled differential equations (3.1) using small time iteration steps $\Delta t$ which are much smaller than damping time in order to obtain the evolution of the beam emittances, relative energy spread and IBS growth times starting from the injected beam emittances.

In our computer code, the numerical integration of the system of Eqs. (3.18–3.20) with replacement given by Eqs. (3.30–3.31) is carried out by Mathematica’s NDSolve function [63] using dynamic programming. Originally our code was developed to use the modified Piwinski formulation (Sec. 3.5) calculating the $f$ integrals along the ring at each iteration. Later a subroutine based on IBS growth times from Bane’s high energy approximation (Sec. 3.3) was developed to compare the results.
Chapter 4

CLIC damping ring lattice

4.1 Initial parameters which drive the design choices

The electron-positron Compact Linear Collider is designed for operation at 3 TeV. Intense bunches injected into the main linac must have unprecedentedly small emittances. The target transverse emittances at the interaction point (IP) of the CLIC main linac must not exceed $\gamma \epsilon_x = 660$ nm in the horizontal and $\gamma \epsilon_y = 10$ nm in the vertical plane in order to achieve the design luminosity $10^{35}$ cm$^{-2}$s$^{-1}$ required for the physics experiments. The positron and electron bunch trains will be provided by the CLIC injection complex. The main beam parameters at the interaction point are given in Table 4.1.

Table 4.1: Beam parameters at the interaction point of CLIC.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Symbol</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bunch population</td>
<td>$N_{bp}$</td>
<td>$2.56 \times 10^9$</td>
</tr>
<tr>
<td>No. of bunches per machine pulse</td>
<td>$N_{bt}$</td>
<td>220</td>
</tr>
<tr>
<td>Repetition frequency</td>
<td>$f_{rr}$</td>
<td>150 Hz</td>
</tr>
<tr>
<td>Bunch spacing</td>
<td>$\tau_b$</td>
<td>8 cm</td>
</tr>
<tr>
<td>Horizontal emittance at IP</td>
<td>$\gamma \epsilon_x$</td>
<td>660 nm</td>
</tr>
<tr>
<td>Vertical emittance at IP</td>
<td>$\gamma \epsilon_y$</td>
<td>10 nm</td>
</tr>
<tr>
<td>RMS bunch length at IP</td>
<td>$\sigma_s$</td>
<td>30.8 $\mu$m</td>
</tr>
</tbody>
</table>

Usually, a positron source produces a bunches with large emittances. The expected upper limit for both horizontal and vertical normalized emittances is $\gamma \epsilon_{x,y} < 50 000 \mu$m. To decouple the wide aperture required for the incoming positron beam from the final emittance requirements of the main linac, an $e^+$ pre-damping ring with a large dynamic acceptance and relatively large equilibrium emittances is needed. In other words, a positron pre-damping ring must reduce the emittance and energy spread of the incoming beam to a low enough values for subsequent injection into the positron main damping ring.

In the case of electron production, taking into account the smaller incoming normalized...
emittance of 7000 nm provided by a high brilliance injector linac, a single ring similar to the main positron damping ring will be sufficient.

Table 4.2: Beam parameters required for the CLIC main damping ring.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Symbol</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bunch population</td>
<td>$N_{bp}$</td>
<td>$2.56 \times 10^9$</td>
</tr>
<tr>
<td>No. of bunches per machine pulse</td>
<td>$N_{bt}$</td>
<td>220</td>
</tr>
<tr>
<td>Repetition frequency (No. of machine pulses per second)</td>
<td>$f_{rr}$</td>
<td>150 Hz</td>
</tr>
<tr>
<td>Horizontal beam emittance at extraction</td>
<td>$\gamma \epsilon_x$</td>
<td>450 nm</td>
</tr>
<tr>
<td>Vertical beam emittance at extraction</td>
<td>$\gamma \epsilon_y$</td>
<td>3 nm</td>
</tr>
<tr>
<td>Longitudinal beam emittance at extraction</td>
<td>$\gamma \sigma_s \sigma_0 c^2$</td>
<td>$&lt; 5000$ eVm</td>
</tr>
</tbody>
</table>

Passing via the bunch compressors, main linac, and beam delivery system the beam emittances increase. To provide the design luminosity at the interaction point, the damping ring complex has to provide intense positron and electron bunch trains with the parameters summarized in Table 4.2. These parameters drive the lattice design of the main damping ring. The rms bunch length $\sigma_s$ and energy spread $\sigma_\delta$ at extraction have to be compatible with the requirement for the subsequent bunch compressors, that is $\gamma \sigma_s \sigma_\delta m_0 c^2 < 5000$ eVm.

A noteworthy feature of the extraction scheme for the positron (electron) CLIC main damping ring is that two trains with 110 bunches separated by 16 cm, are extracted simultaneously from the damping ring with a repetition rate of 150 Hz and these trains need to be combined into a single train using a subsequent delay line and RF deflector. This scheme will be described in detail in Sec. 4.6.3.

The goal of this thesis is to design the optics and performance of the positron main damping ring for the CLIC. The design of the conventional $e^+$ pre-damping ring is not part of the thesis theme. We assume that the design of the NLC positron pre-damping ring [64] with some modification could be adopted to the CLIC injection complex. Also we expect that the positron beam injected to the positron main damping ring will have the parameters listed in Table 4.3.

Table 4.3: Parameters of the beam injected into the CLIC main damping ring.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Symbol</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Horizontal beam emittance at injection</td>
<td>$\gamma \epsilon_x$</td>
<td>$63 \mu$m</td>
</tr>
<tr>
<td>Vertical beam emittance at injection</td>
<td>$\gamma \epsilon_y$</td>
<td>$1.5 \mu$m</td>
</tr>
<tr>
<td>RMS bunch length at injection</td>
<td>$\sigma_s$</td>
<td>10 mm</td>
</tr>
<tr>
<td>RMS relative energy spread at injection</td>
<td>$\sigma_\delta$</td>
<td>0.5%</td>
</tr>
</tbody>
</table>

4.2 TME cell design for the CLIC damping ring

As was mentioned in Sec. 2.8 and Sec. 2.9, we consider a compact four-quadrupole TME cell with short bending magnets and $\mu_x > 180^\circ$. We have designed a TME cell for which the
length $L$ and bending angle $\theta$ of the dipole magnet are 0.545 m and $2\pi/100 = 0.062831$ rad. The structure of the cell is the same as was discussed in Sec. 2.8. For the 2.42 GeV damping ring, the strength of the dipole field produced by this bending magnet is 0.93227 T. We chose the energy of 2.42 GeV. The energy loss per turn for a 2.42 GeV ring consisting of 100 TME cells described above is 0.353 MeV. The transverse damping times $\tau_{x,y}$ are equal to 7.94 ms and the longitudinal damping time $\tau_p = 3.97$ ms. There is no defocusing or focusing field gradient in the bending magnet. The length of the cell is 1.73 m. We set the amplitude of RF voltage to 0.7 MeV.

Using our code based on the modified Piwinski formalism, the equilibrium beam parameters in presence of IBS were computed for this TME cell as a function of horizontal and vertical phase advance. Note that simulations were done at the fixed length of the cell, fixed RF voltage of 0.7 MeV and for a weak betatron coupling of 0.63 %. The change of the phase advance is performed only by the varying the quadrupole strengths. Note that the pair of defocusing quads are identical (equal size and strength) and the pair of focusing quads are identical too.

Figure 4.1 presents the horizontal ($T_x$) and longitudinal ($T_p$) IBS growth times as a function of horizontal and vertical phase advances per the cell. The phase advances $\nu_{x,y}$ are defined in terms of $2\pi$, i.e. $\mu_{x,y} = 2\pi \cdot \nu_{x,y}$. The IBS growth times depend on the lattice functions along the cell and on the equilibrium beam parameters defined by quantum excitation and radiation damping. The average betatron and dispersion function and momentum compaction factor as a function of phase advances $\nu_x$, $\nu_y$ are shown in Fig. 4.2. The comparison between equilibrium beam parameters computed with IBS and without IBS are shown in Fig. 4.3.

As one can see from Fig. 4.3a, the horizontal emittance $\gamma \epsilon_x$ has a minimum at the point $\{\nu_x = 0.625, \nu_y = 0.1\}$ and the longitudinal emittance has a minimum at the point $\{\nu_x = 0.75, \nu_y = 0.1\}$. Nevertheless, the low vertical phase advance of $0.1 \cdot 2\pi$ is not acceptable because of high vertical chromaticity as it could be seen from the Fig. 4.4. For these reasons, to make a compromise between chromaticity and emittance, we chose the phase advances as $\nu_x = 0.584$ and $\nu_y = 0.25$. The lattice functions of this cell are shown in Fig. 4.5 and the parameters of the cell are summarized in Table 4.4. Furthermore, these phase advances allow constructing a second order sextupolar achromat in each arc, which consists of 48 identical TME cell. The first and last cell (50th cell) are used to suppress the horizontal dispersion and do not comprise sextupole magnets.
Figure 4.1: a) the horizontal $T_x$ and b) longitudinal $T_p$ IBS growth times as a function of horizontal $\nu_x$ and vertical $\nu_y$ phase advance per TME cell.
Figure 4.2: The average vertical a), horizontal b), dispersion c) functions and momentum compaction factor d) as a function of phase advances $\nu_x, \nu_y$ per TME cell.
Figure 4.3: The comparison between equilibrium beam parameters computed with IBS and without IBS (denoted by "0") for 100 TME cells.
Figure 4.4: Horizontal $\xi_x$ and vertical chromaticity $\xi_y$ of the TME cell as a function of the betatron phase advance per cell.

Figure 4.5: The lattice functions along the TME cell.
Table 4.4: The parameters of the \( \nu_z = 0.584, \nu_y = 0.25 \) TME cell.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Energy</td>
<td>2.42 GeV</td>
</tr>
<tr>
<td>Field of the bending magnet, ( B_a )</td>
<td>0.932 T</td>
</tr>
<tr>
<td>Length of the bending magnet</td>
<td>0.545 m</td>
</tr>
<tr>
<td>Bending angle</td>
<td>( 2\pi/100 )</td>
</tr>
<tr>
<td>Bending radius</td>
<td>8.67 m</td>
</tr>
<tr>
<td>Length of the cell, ( L_{TME} )</td>
<td>1.73 m</td>
</tr>
<tr>
<td>Horizontal phase advance, ( \mu_x )</td>
<td>210°</td>
</tr>
<tr>
<td>Vertical phase advance, ( \mu_y )</td>
<td>90°</td>
</tr>
<tr>
<td>Emittance detuning factor, ( \epsilon )</td>
<td>1.8</td>
</tr>
<tr>
<td>Horizontal chromaticity, ( \partial \nu_x/\partial \delta )</td>
<td>-0.84</td>
</tr>
<tr>
<td>Vertical chromaticity, ( \partial \nu_y/\partial \delta )</td>
<td>-1.18</td>
</tr>
<tr>
<td>Average horizontal beta function, ( \langle \beta_x \rangle )</td>
<td>0.847 m</td>
</tr>
<tr>
<td>Average vertical beta function, ( \langle \beta_y \rangle )</td>
<td>2.22 m</td>
</tr>
<tr>
<td>Average horizontal dispersion, ( \langle D_x \rangle )</td>
<td>0.0085 m</td>
</tr>
<tr>
<td>Relative horizontal beta function, ( \beta_r = \beta^<em>/\beta_m^</em> )</td>
<td>0.113/0.07 = 1.6</td>
</tr>
<tr>
<td>Relative horizontal dispersion, ( D_r = D^<em>/D_m^</em> )</td>
<td>0.00333/0.00143 = 2.33</td>
</tr>
</tbody>
</table>

The length of our TME cell is quite short which allows getting a small horizontal emittance of \( \gamma \epsilon_{x \theta} = 394 \text{ nm} \) (\( \epsilon_{x \theta} = 8.313 \times 10^{-11} \text{ m} \)). Taking into account IBS, the equilibrium emittance becomes \( \gamma \epsilon_x = 1026 \text{ nm} \). How do the beam parameters depend on the cell length? Let us consider the ring consisting of 100 TME cells with parameters summarized in Table 4.4. The circumference of this ring is equal to 173 m. We studied two variants of the length change: 1) Changing the drift spaces \( s_1, s_2 \) and \( s_3 \) preserving the \( \gamma \epsilon_{x \theta} = 394 \text{ nm} \) and 2) changing the drift spaces together with the length of the bending magnet preserving the \( \gamma \epsilon_{x \theta} = 394 \text{ nm} \). The increase of the length of the magnet is directly proportional to an increase of the drift spaces. The RF voltage is fixed for both variants and equal to 700 kV.

The growth of horizontal equilibrium emittance at presence of IBS is shown in Fig. 4.6a as a function of ring circumference. The dashed lines on the plots correspond to the variant 2 (length of bending magnet is increased together with length of drift space). For the variant 2, the emittance growth due to IBS is stronger than for the variant 1. Figure 4.6b presents the growth of the transverse damping time \( \tau_{x,y} \) with circumference for both variants. The second synchrotron integral \( I_2 \) decreases with an increase of the length of the magnet that is the reason why the damping time in the variant 2 is bigger than in variant 1 where \( \tau \) is just directly proportional to the circumference. The horizontal and vertical chromaticities in the variant 2 linearly change from -84 to -112.7 and from -118 to -112.1, respectively, while the circumference increases from 173 to 356 meters. Furthermore, the main disadvantage of long TME cells is that an increase of the cell length degrades the split of the \( \beta_x \) and \( \beta_y \) functions, which will cause a problem with the chromaticity correction and consequently with the dynamic aperture. For this reason, we designed the TME cell with a very small length of 1.73 m. Further reduction of the cell length is not possible because we must save some space for the sextupoles, BPMs, small dipole correctors and so on. Table 4.5 summarizes the main parameters of the ring consisting of 100 TME cells which were presented in Table 4.4.
and in Fig. 4.5.

Figure 4.6: a) The growth of horizontal equilibrium emittance in the presence of IBS as a function of the ring circumference; b) the growth of the transverse damping time $\tau_{x,y}$ with circumference. The dashed lines on both plots correspond to the case when the length of bending magnet is increased together with the length of drift space. The solid lines correspond to the case when the length of bending magnet is constant and equal to 0.545 m (only drift spaces are changed). The ring consisting of 100 TME cells without wigglers.

Table 4.5: The parameters of the ring consisting of 100 TME cells.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Energy, $E$</td>
<td>2.42 GeV</td>
</tr>
<tr>
<td>Ring circumference, $C$</td>
<td>173 m</td>
</tr>
<tr>
<td>Horizontal emittance w/o IBS, $\gamma\epsilon_{x0}$</td>
<td>394 nm</td>
</tr>
<tr>
<td>Horizontal emittance with IBS, $\gamma\epsilon_{x}$</td>
<td>1026 nm (1100 nm)*</td>
</tr>
<tr>
<td>Horizontal/vertical damping time, $\tau_{x,y}$</td>
<td>7.94 ms</td>
</tr>
<tr>
<td>Horizontal IBS growth time, $T_x$</td>
<td>12.3 ms (11.8 ms)*</td>
</tr>
<tr>
<td>Longitudinal IBS growth time, $T_p$</td>
<td>9.7 ms (9.25 ms)*</td>
</tr>
<tr>
<td>RMS energy spread w/o IBS, $\sigma_\delta$</td>
<td>$7.05 \times 10^{-4}$</td>
</tr>
<tr>
<td>RMS energy spread with IBS, $\sigma_\delta$</td>
<td>$12 \times 10^{-4}$ (12.3 $\times 10^{-4}$)*</td>
</tr>
<tr>
<td>Energy loss per turn, $U_0$</td>
<td>0.353 MeV/turn</td>
</tr>
<tr>
<td>RF frequency, $f_{rf}$</td>
<td>1875 MHz</td>
</tr>
<tr>
<td>Momentum compaction factor, $\alpha_p$</td>
<td>$1.726 \times 10^{-4}$</td>
</tr>
<tr>
<td>RMS bunch length (at $V_{rf} = 700$ kV) w/o IBS, $\sigma_s$</td>
<td>1.2 mm</td>
</tr>
<tr>
<td>RMS bunch length (at $V_{rf} = 700$ kV) with IBS, $\sigma_s$</td>
<td>2.1 mm (2.15 mm)*</td>
</tr>
<tr>
<td>Longitudinal emittance w/o IBS, $\gamma\sigma_s\sigma_\delta m_0 c^2$</td>
<td>2100 eVm</td>
</tr>
<tr>
<td>Longitudinal emittance with IBS, $\gamma\sigma_s\sigma_\delta m_0 c^2$</td>
<td>6045 eVm (6447 eVm)*</td>
</tr>
</tbody>
</table>

* Note that the IBS was computed according to modified-Piwinski method (Sec. 3.5). The values pointed out in the brackets and marked by symbol "**" were computed by Bane’s high energy approximation method (Sec. 3.3).

Note that the parameters in this table were computed for the emittance ratio $\epsilon_{y0}/\epsilon_{x0} = 0.0063$.  

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In order to reach the target emittances stronger radiation damping is needed in order to overcome the effect of IBS. It means that the energy loss per turn has to be largely increased. The most efficient way to increase $U_0$ is to use a damping wigglers with short period.

### 4.3 Change in beam properties due to wigglers

A wiggler magnet is a magnetic device located in a dispersion-free straight section of the damping ring. A wiggler magnet produces a vertical field which alternates in polarity along the beam direction. In general, wiggler magnets give rise to both radiation damping and quantum excitation, and so they result in different equilibrium values of damping times, emittance and energy spread which depend both on the wiggler magnet parameters and on the lattice functions through the wiggler.

A wiggler is a periodic magnet system. In the first order approximation, the vertical field component $B_y$ of a wiggler raises along the beam axis as

$$B_y = B_w \sin \left( \frac{2\pi s}{\lambda_w} \right)$$

where $B_w$ and $\lambda_w$ are the peak field on the beam axis and the wiggler period length, respectively. Such field distribution can be produced if the wiggler period $\lambda_w$ consists of: a drift space of length of $\lambda_w/8$ → magnetic pole with length of $\lambda_w/4$ producing positive vertical dipole field → drift space of length of $\lambda_w/4$ → magnetic pole with length of $\lambda_w/4$ producing negative vertical dipole field → drift space of length of $\lambda_w/8$.

The contribution from a wiggler to the $i^{th}$ synchrotron integrals can be written as

$$I_i = I_{ia} + I_{iw}$$

where $I_{ia}$ and $I_{iw}$ are the synchrotron integrals produced in the arcs and in the wigglers respectively.

Assuming that wiggler magnets with sinusoidal field variation are installed in the dispersion-free region of the machine, the integrals $I_{iw}$ can be written as follows [65]:

$$I_{2w} = \frac{L_{ID}}{2\rho_w^3}, \quad I_{3w} = \frac{4}{3\pi} \frac{L_{ID}}{\rho_w^3}, \quad I_{4w} = \frac{3}{32\pi^2} \frac{\lambda_w^2}{\rho_w^5} L_{ID}, \quad I_{5w} = \frac{\lambda_w^4}{4\pi^4\rho_w^5} \left[ \frac{3}{5\pi} + \frac{3}{16} \right] \langle \gamma_x \rangle L_{ID} - \frac{9\lambda_w^3}{40\pi^4\rho_w^5} \langle \alpha_x \rangle L_{ID} + \frac{\lambda_w^2}{15\pi^2\rho_w^5} \langle \beta_x \rangle L_{ID}$$

Here, $\rho_w$ is the bending radius of the wiggler magnet, the operator $\langle \ldots \rangle$ denotes the average of the horizontal Twiss parameters $\alpha_x, \beta_x, \gamma_x$ through the wiggler of length $L_{ID}$. The length of the wiggler magnet, $L_{ID}$, is equal to $\lambda_w \cdot N_p$ where $N_p$ is the number of periods per one wiggler magnet. The $I_{4w}$ and $I_{5w}$ terms arise from the dispersion generated by the wiggler magnet itself, the so-called self-dispersion. In most cases, the term $I_{4w}$ is negligible compared to the larger $I_{2w}$ term.

For a hard-edged wiggler field model (rectangular field model) where dipoles with field of $B_w$ occupy half of the wiggler length (i.e., a filling factor of 50 %), the synchrotron integral $I_{2w}$ is same as for the sinusoidal field model. For the hard-edged and sinusoidal field models, the synchrotron integral $I_{5w}$ generated by the wiggler is slightly different. Assuming that $\langle \alpha_x \rangle$ is small value through the wiggler, the largest dominant terms of $I_{5w}$ for both models
are written as

\[ I_{5w} = \frac{\lambda_w^2}{15\pi^3 \rho_w^5} \langle \beta_x \rangle L_{ID} \quad I_{5w} = \frac{\lambda_w^2}{384 \rho_w^5} \langle \beta_x \rangle L_{ID} \] (4.4)

sinusoidal field models hard - edged field model

The value of \( I_{5w} \) derived from hard-edged field approximation is bigger by factor 1.21 than the value of \( I_{5w} \) in sinusoidal field representation. The integral \( I_{3w} \) in the hard-edged field approximation becomes \( I_{3w} = L_{ID}/(2\rho_w^3) \). The maximum dispersion in the wiggler period is

\[ D_{w}^{max} = \frac{3 \lambda_w^2}{64 \rho_w} \]

and the bending radius \( \rho_w \) is given as

\[ \rho_w = \frac{(B\rho)}{B_w} \quad \text{or} \quad \rho_w[m] = \frac{0.0017\gamma}{B_w[T]} \] (4.5)

where \( (B\rho) \) is the standard energy dependent magnetic rigidity.

Though the sinusoidal field representation is more realistic, in the further discussion, we will use the hard-edged field approximation because for wigglers with a short period the difference between the two models is small. Moreover, we use the MAD [79] code to design the linear optics for the CLIC damping ring. In this code wigglers are approximated by a hard-edged model. The nonlinearities and high order field components will be studied in details in the later Chapter 6 "Nonlinearities induced by the short period NdFeB permanent wiggler and their influence on the beam dynamics".

In the linear optics approximation in order to compute the change of the beam properties due to introducing a wiggler magnet, we need to know the wiggler period \( \lambda_w \), peak field \( B_w \) and total length of the wigglers \( L_w (= N_w \cdot L_{ID} \) where \( N_w \) is the total number of the wiggler magnets in the ring). The change of the damping rate due to the wiggler is conventionally defined by the relative damping factor that is

\[ F_w \equiv \frac{I_{2a}}{I_{2a}} = \frac{L_w B_w^2}{4\pi (B\rho) B_a} = \frac{1}{4\pi \cdot 0.0017 [\text{Tm}] \gamma B_a} \geq 0 \] (4.6)

where \( B_a \) is the field of the bending magnets. When \( F_w > 1 \), the damping is dominantly achieved in the wigglers. The energy loss per turn is

\[ U_0 = U_{0a}(1 + F_w) = 3.548 \times 10^{-12}[\text{MeV}] \gamma^3 B_a[T](1 + F_w) \]

where \( U_{0a} \) is the energy loss produced only in the arcs that is given by Eq. (2.28). The damping partition can be expressed as

\[ J_x = \frac{J_{xa} + F_w}{1 + F_w} \] (4.7)

where \( J_{xa} \) is the contribution from the arc cells alone that is given by Eq. (2.57). The damping partition \( J_{xa} \) can be decreased using combined function bending magnets in the
arcs, however, when $F_w \gg 0$, the fractional change in $J_x$ becomes smaller. The radiation damping times are

$$\tau_x = \frac{2E_0T_0}{J_xU_0} = \frac{3(B\rho)C}{2\pi r_0 c\gamma^3 B_a(J_{xa} + F_w)} = \frac{E_2}{B_a \gamma^2 (J_{xa} + F_w)}$$

(4.8)

$$\tau_y = \frac{2E_0T_0}{J_yU_0} = \frac{3(B\rho)C}{2\pi r_0 c\gamma^3 B_a(1 + F_w)} = \frac{E_2}{B_a \gamma^2 (1 + F_w)}$$

(4.9)

$$\tau_p = \frac{2E_0T_0}{J_sU_0} = \frac{3(B\rho)C}{2\pi r_0 c\gamma^3 B_a(3 - J_{xa} + 2F_w)} = \frac{E_2}{B_a \gamma^2 (3 - J_{xa} + 2F_w)}$$

(4.10)

where the constant $E_2$ is

$$E_2 = \frac{3(B\rho)}{2\pi r_0 c\gamma} = \frac{3 \cdot 0.0017 \text{ [Tm]}}{2\pi r_0 c} = 960.13 \text{ [T \cdot sec \cdot m]}$$

Before choosing a value for $F_w$, the effect of the wiggler on other parameters, including emittance and momentum compaction, must be considered. By expanding the $I_2$ and $I_5$ synchrotron integrals according to Eq. (4.1) and using Eqs. (4.2), (4.4) for the hard-edged model, (4.6) and (4.7), the horizontal equilibrium emittance given by Eq. (2.20) is written as

$$\gamma\epsilon_{x0} = \frac{C_q \gamma^3}{12 (J_{xa} + F_w)} \left[ \epsilon_r^3 \theta^3 + \frac{F_w|B_w^3|\lambda_{w}^3(\beta_x)}{16(B\rho)^3} \right]$$

(4.11)

This approximation ignores the details of the dispersion suppressor optics at the start and end of the arcs, but is still a fairly accurate description, especially when the number of TME cells per arc is large (e.g., > 10).

The equilibrium rms relative energy spread $\sigma_\delta$ given by Eq. (2.45) is rewritten as

$$\sigma_\delta = \gamma \sqrt{\frac{C_q I_3}{2I_2 + I_3}} = \gamma \left[ \frac{1 + F_w}{B_a} \frac{|B_a|}{(B\rho)} \frac{|B_w^3|\lambda_{w}^3(\beta_x)}{3 - J_{xa} + 2F_w} \right]^{1/2}$$

(4.12)

and rms bunch length $\sigma_s_0$ (for zero current) are given by Eq. (2.46). The equilibrium bunch length depends on the $\alpha_p$ momentum compaction and the parameters of the RF system. The momentum compaction defined by Eq. (2.36) can be expressed as [67]

$$\alpha_p = \frac{3\pi}{2} \left( \frac{4\sqrt{15}}{9} \right)^{2/3} \frac{(B\rho)(1 + F_w)^{2/3}}{C|B_a| \gamma^2} \times \left( \frac{\gamma\epsilon_{x0}}{C_q} - \frac{|B_w^3|\lambda_{w}^3(\beta_x)\gamma^3}{192(B\rho)^3} \frac{F_w}{J_{xa} + F_w} \right)^{2/3} \times \frac{\sqrt{5} + \sqrt{\epsilon_r^2 - 1}}{\epsilon_r^{2/3}}$$

Note that emittance $\gamma\epsilon_{x0}$ is defined by Eq. (4.11) rather than Eq. (2.56) as it was before. The lattice design has to provide a relatively large momentum compaction of the ring to avoid instability thresholds and to reduce the sensitivity to circumference changes.
4.4 Lattice design of the wiggler FODO cell

The average horizontal beta function \( \langle \beta_x \rangle \) through the wiggler is to be much larger than the value of \( \frac{\lambda_w}{2\pi} \). The wigglers can produce either an increase or decrease of the equilibrium emittance with respect to the value of the emittance produced in the arcs. It depends on the relationship between wiggler parameters, \( \langle \beta_x \rangle \) and emittance in the arcs. From Eq. (4.11) one can derive the condition under which the beam emittance is unperturbed or reduced:

\[
\lambda_w^2 \leq 5.87 \times 10^9 \frac{E [\text{GeV}] \epsilon_{a_0}}{B_w^2 \langle \beta_x \rangle}
\]  

(4.13)

with \( E \) the beam energy in units of GeV and \( B_w \) in units of Tesla. The emittance \( \epsilon_{a_0} \) denotes the value of emittance generated in the arcs. The period length \( \lambda_w \) must not exceed the value determined by Eq. (4.13) in order to obtain a reduced emittance.

The mean beta function \( \langle \beta_x \rangle \) through the wiggler can be kept reasonably small. By using a FODO lattice [65] to construct a dispersion-free straight section for the placement of a wiggler magnets, the value of \( \langle \beta_x \rangle \) is approximately twice the length between quadrupoles in the FODO cell with phase advances of \( \mu_x = \mu_y \sim 90^\circ \). A FODO cell containing two wiggler magnets which occupy the space between quadrupoles will be considered. On the one hand it is useful to keep the length of the FODO cell relatively short but on the other hand, a large number of short wiggler magnets, as a consequence of a short FODO cell, may generate significant nonlinearities due to the fringe field. In addition, for a fixed length of the straight section, a series of short FODO cells generates a larger value of chromaticity in comparison with long FODO cell.

Taking into account conventional designs for strong wigglers (\( B_w \sim 2\, \text{T} \)) with short period (\( \lambda_w < 20\, \text{cm} \)), we will consider a damping wiggler with length of \( L_{ID} = 2\, \text{m} \). The average horizontal beta function \( \langle \beta_x \rangle \) and chromaticities \( \xi_x, \xi_y \) as a functions of the horizontal and vertical phase advance \( \nu_x, \nu_y \) of the FODO cell with length of 4.6 m are shown in Fig. 4.7. Note that the scan shown in Fig. 4.7 was done for two wiggler magnets (\( \lambda_w = 10\, \text{cm}, \ \ B_w = 1.7\, \text{T} \) and \( L_{ID} = 2\, \text{m} \)) in the FODO cell.

In the CLIC damping ring design, we considered the FODO cell with phase advance of \( \mu_x = 0.26 \times 2\pi = 93.6^\circ \) and \( \mu_y = 0.24 \times 2\pi = 86.4^\circ \). These phase advances give high flexibility for the phase tuning between arcs and FODO straight section as will be discussed below. The lattice functions of the 93.6°/86.4° FODO cell are shown in Fig. 4.8. Note that the right-side axis for the horizontal dispersion \( D_x \) is shown in units of millimeters. For this wiggler FODO cell, the chromaticities and average horizontal and vertical beta functions amount to \( \xi_x = -0.313, \ \xi_y = -0.29 \) and \( \langle \beta_x \rangle = 3.7\, \text{m}, \ \langle \beta_y \rangle = 3.9\, \text{m} \).
Figure 4.7: The average horizontal beta function $\langle \beta_x \rangle$ and chromaticities $\xi_x$, $\xi_y$ as a functions of the horizontal and vertical phase advance $\nu_x$, $\nu_y$ of the FODO cell with length of 4.6 m.

Figure 4.8: The lattice function of the $\mu_x = 0.26 \times 2\pi = 93.6^\circ$, $\mu_y = 0.24 \times 2\pi = 86.4^\circ$ FODO cell with two wiggler magnets ($\lambda_w = 10$ cm, $B_w = 1.7$ T and $L_{ID} = 2$ m).

A wiggler magnet introduce a vertical betatron tune shift $\Delta \nu_y$ that can be estimated as

$$\Delta \nu_y = \frac{\langle \beta_y \rangle L_{ID} N_w}{8\pi \rho_w^2} = \frac{\langle \beta_y \rangle L_{ID} N_w B_w^2}{8\pi (B\rho)^2} \quad (4.14)$$
The vertical tune shift produced by a pair of wigglers \((L_{ID} = 2 \text{ m})\) with peak field of 1.7 T is equal to 0.0275. A pair of wigglers with the same length but with field of 2.52 T give the vertical tune shift of 0.06. However, by adjusting the strength of the focusing and defocusing quadrupoles in the range of \((3.24 \div 3.13) \text{ m}^{-2}\) and \((-3.1 \div -2.55) \text{ m}^{-2}\), respectively, it is possible to maintain the fixed phase advances \(\mu_x = 0.26 \times 2\pi, \mu_y = 0.24 \times 2\pi\) per the FODO cell, for the wigglers with peak field from 0 T to 2.52 T.

### 4.5 Lattice design of the dispersion suppressor and beta-matching section

As it was discussed above, we chose a racetrack design of the CLIC damping ring. Let us consider the TME cell listed in Table 4.4 starting from the middle of the bending magnet to the middle of the subsequent bending magnet. In this case, each arc consists of 48 TME cells (with sextupoles) plus 2 TME-like cells in the ends of the arc, so-called dispersion suppressor cells, which are used to suppress the horizontal dispersion in the straight sections.

It is easy to construct a dispersion suppressor based on the TME cell listed in Table 4.4, if the last TME cell is terminated by a bending magnet having a field integral two times smaller than the field integral of a bending magnet in the arc. In our design for the dispersion suppressor, a half length bending magnet with the field of 0.932 T is used. The lattice functions through the dispersion suppressor are shown in Fig. 4.9.

![Figure 4.9: The lattice structure of the beta-matching section followed by dispersion suppressor.](image-url)
The drift lengths and quadrupole strengths in the dispersion suppressor are slightly modified with respect to the drift lengths and quadrupole strengths in the TME cell. Four quadrupoles in the dispersion suppressor must have an independent power supply so as to enhance flexibility for precise adjusting of $D_x$, $D'_x$ to zero at the end of the half length bend and also to assist in matching of the betatron functions to the FODO cell.

However, four quadrupoles are not sufficient to precisely adjust the betatron functions to the FODO cell. Thus, a short and adjustable (for phase advance) beta-matching section, which separates the FODO straight section from the dispersion suppressor, is needed.

A beta-matching section allows to fit the beta functions from the suppressor to the FODO cell and also to adjust the phase advance between the arc and the straight section. The lattice design of the beta-matching section followed by dispersion suppressor is shown in Fig. 4.9. Tuning eight independent quadrupoles located in the suppressor and beta-matching section, provides a precise beta-matching to FODO cell and exact compensation of the dispersion. Moreover, at the same time, it is possible to vary the horizontal and vertical phase advance between the last TME cell and the first FODO cell with wigglers $(0 < B_w < 2.52 \text{T})$ are in the range of $\mu_x = (0.26 \pm 0.01) \times 2\pi$ and $\mu_y = (0.24 \pm 0.01) \times 2\pi$. This offers the possibility to adjust the horizontal and vertical phase advance between arcs to an integer number of $2\pi$ if the number of the FODO cells with wigglers is larger than 10 cells. In other words, we can always establish a $+I$ transformation matrix (see Eq. 2.11) between arcs if the straight section consists of at least 10 FODO cells. This possibility is very helpful for further nonlinear optimization of the lattice.

Three beta-matching sections, as shown in Fig. 4.9, are used for the damping ring. Two of these beta-matching sections include a wiggler magnets but the third section does not, because the equivalent space is reserved for RF superconducting cavities needed to compensate the energy loss.

To inject and extract bunch trains, a septum magnets and kickers are needed for the damping ring. For this reason, an injection/extraction insertion between regular FODO cell and dispersion suppressor must also be designed. It is presented in the next section.

The Twiss functions at the end of the last bend are $\alpha_x = -1.0$ and $\beta_x = 0.23 \text{ m}$. Taking into account Eq. (2.49), (2.59) and (2.61), the difference from the optimal values are $\Delta \alpha_0 = -2.87$ and $\beta_r = \beta_r/\beta_{opt} = 0.54$. Due to the last magnets in the arcs (suppressor magnets), disturbance of the horizontal emittance $\gamma \epsilon_{x0}$ that is produced only by the arc bending magnets (without wigglers and IBS) is less than 5 %.

A change of the horizontal phase advances per the FODO cell in the limits of $(0.26 \pm 0.01) \times 2\pi$ results in a change of the average horizontal beta function through the FODO cell in the limits of $3.7 \pm 0.07 \text{ m}$. The change in equilibrium emittance (without IBS) $\Delta \gamma \epsilon_{x0}$ does not exceed 3 nm if the $\langle \beta_x \rangle$ is changed in the range of $3.7 \pm 0.07 \text{ m}$ through the FODO straight sections which include 76 units of wiggler magnets with parameters of $L_{ID} = 2 \text{ m}$, $1.5 \text{T} < B_w < 2.5 \text{T}$ and $\lambda_w < 0.1 \text{ m}$. Note that this estimate was done for the optimal values of $\lambda_w$ corresponding to each particular value of $B_w$, which will be defined in Sec.4.7.1.

Therefore, assuming a large number of the wiggler magnets, the ring tunes can be easily varied over a wide range without any significant variation in the horizontal emittance.
4.6 Injection and extraction

4.6.1 Lattice design of the injection/extraction region

Single-turn injection in the horizontal plane is considered for the CLIC damping ring. The bunch train from the positron pre-damping ring passing through a beam transfer line is brought onto the orbit of the main damping ring by using a septum magnet and a fast kicker element.

The first requirement for the injected beam is that at the exit of the septum (end of the beam transfer line), the betatron and dispersion function $\beta_x$, $\beta_y$, $\alpha_x$, $\alpha_y$, $D_x$, and $D'_x$ must be identical with the ring lattice parameters at this point. The quadrupoles in the transfer line are then used to match the beam ellipses. Also, at the exit of the septum, the injected beam must be at a horizontal distance $x_{sep}$ from the center of the machine aperture

$$x_{sep} \geq N_x (\sigma_{xi} + \sigma_{xs}) + D_x \sigma_{pi} + \langle x_i \rangle + \langle x_c \rangle + d_{sep} \quad (4.15)$$

where $\sigma_{xi}$ and $\sigma_{xs}$ are the rms beam sizes of the incoming beam and of the stored beam, respectively, $N_x$ is the distance between the closed orbit and the septum plate in units of the injected beam size (the choice of this number depends on the dynamic aperture, e.g., reasonable values may be $N_x \geq 7$ for electron rings), $\sigma_{pi}$ the rms relative momentum deviation of the injected beam, $\langle x_i \rangle$ the rms orbit variation of the injected beam in the septum magnet, $\langle x_c \rangle$ the rms closed-orbit offset at the location of the septum, and $d_{sep}$ the thickness of the septum.

The injected beam must be at the center of the aperture when it reaches the kicker. In this case, the condition $x_{kic} = R_{11} x_{sep} + R_{12} x'_{sep} = 0$ determines the correlation of angle $x'_{sep}$ and offset $x_{sep}$ of the injected beam at the exit of the septum:

$$x'_{sep} = -\frac{\alpha_{sep} + \cot \Delta \mu}{\beta_{sep}} x_{sep} \quad (4.16)$$

where $R$ and $\Delta \mu$ denotes the $2 \times 2$ transport matrix (2.11) and phase advance between the septum and the the kicker, respectively. The angle can be adjusted by changing the strength of the septum magnet. The kicker must then apply an angular deflection of

$$\theta_{kick} = -\frac{x_{sep}}{\sin \Delta \mu \sqrt{\beta_{sep} \beta_{kick}}} \quad (4.17)$$

A large value of $\beta_{kick}$ reduces the kicker strength, and a large $\beta_{sep}$ value also reduces the relative contribution to $\theta$ due to the septum thickness.

It is clear that a phase advance of $\pi/2$ from septum to kicker will effectively convert an amplitude at the septum to an angular kick at the kicker. The defocusing quadrupoles between the septum and kicker aid by inflecting the trajectory of the injected beam to the reference closed orbit.
Taking into account the above consideration, a lattice design of the injection and extraction region of the CLIC damping ring was developed as shown in Fig. 4.10. The length of the injection/extraction section is exactly equal to the length of the beta-matching section shown in Fig. 4.9. The septum magnet IN-SEP and pulsed kicker magnet IN-KICKER are used for the injection. The septum magnet EX-SEP and pulsed kicker magnet EX-KICKER are used for the extraction. The kickers IN-KICKER and EX-KICKER are placed just up-stream and down-stream, respectively, of the F-quadrupoles where the beta function is largest. The phase advance between IN-SEP and IN-KICKER is $\pi/2$. The phase advance between EX-KICKER and EX-SEP is the same.

The positions and length of all elements in the dispersion suppressor remain unchanged, but the strength of the quadrupoles is slightly modified. Two FODO cells with wigglers following the injection/extraction section must have independent power supplies of the quadrupoles in order to adjust the lattice functions to the regular FODO cell. The horizontal and vertical phase advance through the dispersion suppressor, injection/extraction section and two adjustable FODO cells are equal to $\mu = 1.68 \times 2\pi$ and $0.82 \times 2\pi$. Moreover, the horizontal and vertical phase advances through the beta-matching section together with dispersion suppressor (see Fig. 4.9) and two regular FODO cells are identical. Consequently, the phase advances between the two arcs are identical. Additional tune shifts across both long straight sections can also be introduced, if desirable.
4.6.2 Requirements for the septum and kicker magnets

We assume that a DC septum magnet has to be used in the CLIC damping ring because a pulsed septum may cause more jitter problems than a DC septum. The design parameters of the septum magnet are based on the septum design developed for the NLC damping ring [68], but the strength of the dipole field was scaled to the ring energy of 2.42 GeV and to the required effective length of 0.9036 m. Moreover, the thickness of the second blade is reduced from the 15 mm to 13 mm. The thickness of the first blade remains unchanged and equal to 5 mm.

The septum magnet consists of two sections -SEP-1 and -SEP-2 which have a different blade thickness (namely 5 mm and 13 mm, respectively, as already mentioned) and consequently different strength of the magnetic field. The maximum strength of the dipole field produced by a septum magnet is limited by the thickness of the blade. The septum blade (sometimes called knife or plate) in a DC septum magnet with relatively strong field cannot be made much thinner than 3 mm [69].

The Twiss parameters at the entrance of the injection septum (section IN-SEP-1) are the following: \( \beta_{sep} = 3.8 \) m, \( \alpha_{sep} = 2.04 \) and \( D_x = 0 \). For reliable injection, the horizontal distance between the injected beam trajectory after the septum magnet and the edge of the blade must be larger than \( \langle x_i \rangle = 2.5 \) mm. Assuming \( N_x = 15 \) and \( \langle x_c \rangle = 2 \) mm in Eq. (4.15), the horizontal distance between injected trajectory and the design orbit in the ring, \( x_{sep} \), at the septum with blade thickness \( d_{sep} = 5 \) mm has to be larger than 13 mm for \( \gamma e^{\text{inj}} = 63 \) \( \mu \)m.

In our design, the average horizontal beta function along the kicker IN-KICKER is \( \beta_{kick} = 10.7 \) m. The phase advance from the exit of septum to the center of the kicker is \( \Delta \mu = 92^\circ \). Commonly used ferrite kickers operate with voltage and current levels of 80 kV and 5000 A, and with fields of 500 Gauss.

In our damping ring design, the ferrite kicker has a length \( l = 0.4 \) m. From the well known relation

\[
\theta_{\text{mrad}} = \frac{29.98}{E_{\text{GeV}}} (Bl)_{\text{kG-m}} \tag{4.18}
\]

the maximum angular kick produced by this ferrite kicker is 2.5 mrad for a field strength of 500 Gauss.

The kicker magnets must be fast. Namely, the rise and fall times of the kicker magnetic field must be less than the gap between bunch trains. According to the CLIC design specification and as further discussed in Sec. 4.6.3, two bunch trains must be simultaneously extracted or injected during a single kicker pulse. The kicker field must be flat for the duration of the two bunch trains including the gap between them. Therefore, the kickers for the CLIC damping ring must provide short rise and fall times of 25 ns with a 142 ns long flat top (flat field region). The design for such kicker may be based on the performance of the ferrite kicker at the ATF damping ring [70].

Figure Fig. 4.11 shows the injected and extracted beam trajectories through the septum magnets. The requirements for the septum magnets and kickers are listed in Tables 4.6 and 4.7, respectively. The tolerance on the pulse-by-pulse deflection error listed in Table 4.7 for the injection and extraction kickers corresponds to a centroid jitter of 0.1\( \sigma_x \) for the injected and extracted horizontal beam size, respectively.
Figure 4.11: The injected and extracted beam trajectories through the septum. The horizontal phase advance is pointed out by the black curve.

The extraction is accomplished by the analogous magnet components but in the reverse order; first a deflection by the kicker, then a deflection by the septum. Since the lattice of the injection/extraction region has a mirror symmetry, the parameters and requirements for the EX-SEP and EX-KICKER are similar to those for the IN-SEP and IN-KICKER. However, the tolerance on the deflection error for the extraction kicker is about ten times tighter than that for the injection kicker, due to the smaller size of the damped beam.

Table 4.6: Parameters of the septum magnets

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Septum -SEP-1</th>
<th>Septum -SEP-2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Effective length</td>
<td>0.4018 m</td>
<td>0.5018 m</td>
</tr>
<tr>
<td>Bending angle</td>
<td>13 mrad</td>
<td>42 mrad</td>
</tr>
<tr>
<td>Field integral</td>
<td>0.105 T·m</td>
<td>0.339 T·m</td>
</tr>
<tr>
<td>Blade thickness</td>
<td>5 mm</td>
<td>13 mm</td>
</tr>
<tr>
<td>Type</td>
<td>DC</td>
<td>DC</td>
</tr>
</tbody>
</table>

4.6.3 Injection and extraction scenario

To achieve the peak design luminosity $10^{35}\text{cm}^{-2}\text{s}^{-1}$, both electron and positron bunch trains consisting of 220 bunches with separation between bunches of 8 cm (0.267 ns) must collide in
Table 4.7: Parameters of both kickers

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rise time</td>
<td>$\leq 25$ ns</td>
</tr>
<tr>
<td>Fall time</td>
<td>$\leq 25$ ns</td>
</tr>
<tr>
<td>Flat top</td>
<td>142 ns</td>
</tr>
<tr>
<td>Repetition rate</td>
<td>150 Hz</td>
</tr>
<tr>
<td>Beam energy</td>
<td>2.42 GeV</td>
</tr>
<tr>
<td>Effective length</td>
<td>0.4 m</td>
</tr>
<tr>
<td>Angular deflection</td>
<td>2.45 mrad</td>
</tr>
<tr>
<td>Field</td>
<td>500 Gauss</td>
</tr>
<tr>
<td>Beta at kicker</td>
<td>10.7 m</td>
</tr>
<tr>
<td>Core material</td>
<td>Ferrite TDK</td>
</tr>
<tr>
<td>Injection kicker tolerance</td>
<td>$\pm 1.44 \times 10^{-3}$</td>
</tr>
<tr>
<td>Extraction kicker tolerance</td>
<td>$\pm 1.5 \times 10^{-4}$</td>
</tr>
</tbody>
</table>

the interaction point with the repetition rate $f_{rr} = 150$ Hz. Such positron (electron) bunch trains with the required beam emittances have to be extracted from the CLIC damping rings with the same repetition rate 150 Hz. The time between two subsequent trains is called the machine pulse and is equal to $6.66 \text{ ms} (= f_{rr}^{-1})$.

A noteworthy feature of the extraction scheme for the positron (electron) CLIC damping ring is that two trains with 110 bunches separated by 16 cm, are extracted simultaneously and need to be combined using a subsequent delay line and RF deflector. In other words, two bunch trains separated by a gap of $25.6 \text{ ns}$ ($48 \times 0.16 \text{ m}$) are extracted simultaneously during one pulse of the kicker EX-KICKER. The injection and extraction scheme of the CLIC damping ring with double kicker system is illustrated in Fig. 4.12a. As was mentioned above, the kicker must provide a flat top of 142 ns with rise & fall times shorter than 25 ns.
The advantage is a two times larger bunch spacing in the damping ring (16 cm) compared with the main linac (8 cm), which alleviates the impact of electron-cloud and fast-ion instabilities, and allows for a lower RF frequency in the damping ring, which also leads to a longer bunch and reduces the effect of intrabeam scattering.
The stable beam extraction from the damping ring is essential for the linear collider to achieve high luminosity. Thus, it is extremely important that the extraction kicker has a very small jitter which refers not only to the uniformity of the pulsed magnetic field but also to its pulse-to-pulse stability. To reduce the jitter tolerance of the extraction kicker EX-KICKER, the double kicker system for the CLIC damping ring is suggested.

The system uses two identical kicker magnets separated by the phase advance $(2n + 1)\pi$. The first kicker EX-KICKER is placed in the damping ring and the second kicker EX-KICKER-2 for jitter compensation in the extraction line (see Fig. 4.12a). Both kickers have a common pulse power supply as shown in Fig. 4.12b and produce a kick in the horizontal plane. The two transmission cables from the pulser to the kickers have different lengths because of the beam travel time delay between the kicker magnets.

If both kickers have a kick angle variation $\Delta \theta_1$ and $\Delta \theta_2$, then the coordinates $(x, x')$ at the exit of the second kicker can be written as

$$
\begin{pmatrix}
x \\
x'
\end{pmatrix} = M_{1 \to 2} \begin{pmatrix} 0 \\ \Delta \theta_1 \end{pmatrix} + \begin{pmatrix} 0 \\ \Delta \theta_2 \end{pmatrix}
$$

Here, $M_{1 \to 2}$ is a transfer matrix given by Eq. (2.11) from the first kicker to the second one. Since the phase advance between the two kickers is $\pi$, Eq. (4.19) is simplified to

$$
x = 0, \quad x' = -\sqrt{\frac{\beta_1}{\beta_2}} \Delta \theta_1 + \Delta \theta_2
$$

When the two kickers are identical, that means $\Delta \theta_1 = \Delta \theta_2$, the variation could be cancelled with the same beta functions. If the two kickers are not identical, compensation also can be achieved by adjusting the $\beta_2$ function. However, the system becomes more sensitive to timing jitter if the two kickers do not have identical waveforms.

To separate the first and the second bunch train, the kicker EX-KICKER-3 is used in the extraction line. It must provide a flat top of 58.13 ns $(109 \times 0.533)$ with rise & fall times better than 25 ns in order to deflect the first train into the delay line. It does not act to the second train which passes through the straight line. Since the gap between trains is 25.6 ns, the pass-length of the delay line (pass-length between EX-KICKER-3 and RF deflector) must be longer than length of the straight line by $(109 \times 0.5333 + 25.6 - 0.5333/2) = 83.46$ ns or 25.04 m. In this case, the first train bunches are interleaved with the second train bunches at the RF deflector as shown in Fig. 4.12c. The RF deflector [71] is made from short resonant, traveling-wave, iris-loaded structures with a negative group velocity. The bunches arriving from the delay line receive a positive kick. The bunches coming from the straight line receive the negative kick of the same strength. The two trains are now combined into one single train. This train consists of 220 bunches with a spacing of 0.266 ns.

The injection into the damping ring is straightforward. Two bunch trains with a gap of 25.6 ns are injected simultaneously during one pulse of the kicker IN-KICKER. Since the trajectories of the extracted and injected trains are crossing, if injection and extraction are performed from the same side of the beamline, the injection and extraction cannot be accomplished simultaneously on the same turn. The RF buckets which have been occupied by the extracted bunches must be then filled by injected bunches on the next turn. However, to keep the beam loading of the cavity in the damping ring almost constant, the RF buckets which have been occupied by the just-extracted bunches should be filled immediately by injected bunches. Such simultaneous transfer could be possible, if the injection is performed from the other side of the injection/extraction section, in which case the blade of the septum IN-SEP should be located at $(x_{sep} < 0)$. 

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4.7 Beam properties for the racetrack design of the CLIC damping ring

4.7.1 Beam properties without the effect of IBS

In this section, beam properties are considered without the effect of IBS. The ring circumference $C$ can be expressed as a sum $96L_{TME} + 4(L_{supp} + L_{ms}) + L_w - 4L_{ID} + L_{fdo}$, where $L_{supp}$ and $L_{ms}$ are the lengths of the dispersion suppressor and the beta-matching section, respectively, and $L_{ID} = 2$ m the length of a single wiggler (insertion device). Note that the length of the injection/extraction section is exactly equal to the length of the beta-matching section. The $L_{fdo}$ is the total length of the FODO quads and FODO drifts which are not occupied by wigglers. Let us denote the sum $4(L_{ms} - L_{ID}) + L_{fdo}$ as $\Delta L$. It is equal to $17.55$ m $+ (N_w - 4) \cdot 0.3$ m where $N_w$ is a total number of wigglers in the ring. For the TME cell listed in Table 4.4, the length of the arcs with the dispersion suppressors is $L_a = 96L_{TME} + 4L_{supp} = 173.78$ m. Therefore, the circumference of the damping ring is $C = L_a + \Delta L + L_w = 191.34 + 0.3(N_w - 4) + 2N_w$ in units of meters, assuming that a single wiggler module is 2 m long.

The choice of $F_w$ impacts many other critical ring parameters such as the momentum compaction and the beam energy spread, as described in Sec. 4.3. Figure 4.13a shows the relation between wiggler peak field $B_w$ and total length of wigglers $L_w$ for constant values of the relative damping factor $F_w$. Figure 4.13b shows the dependence of relative energy spread $\sigma_\delta$ (without IBS) on $B_w$ and $L_w$.

![Figure 4.13](image)

Figure 4.13: a) The relation between wiggler peak field $B_w$ and total length of wigglers $L_w$ for constant values of the relative damping factor $F_w$; b) dependence of relative energy spread $\sigma_\delta$ (without IBS) on $B_w$ and $L_w$.

The relative energy spread $\sigma_\delta$ does not depend on the wiggler period $\lambda_w$. In the range $B_w < 1$ T, the $\sigma_\delta$ does not increase with $L_w$ as shown in Fig. 4.13b.
However, the growth of $\sigma_{00}$ with $B_w$ becomes approximately linear if the wiggler peak field is increased in the range $B_w > 1$ T. The damping time given by Eqs. (4.8–4.10) is also independent of the wiggler period $\lambda_w$. As shown in Fig. 4.14, the damping time $\tau_{x0}$ for the damping ring design given above reaches an asymptotic value for $F_w \gg 1$, if the total length of the wigglers is larger than

$$L_w \approx \frac{L_a + 4(L_{ms} - L_{ID}) - 1.2[m]}{\frac{\gamma \tau_{x0} \gamma^3 B_w^2}{6(Bp)^2} \frac{1+F_w}{F_w} - 1.15} = \frac{191.34}{0.04875 B_w^2 [T] \gamma \tau_{x0} [s] - 1.15} \quad (4.21)$$

In this case, the wiggler does all the damping and the arcs do none. Note that expression (4.21) gives an overestimated (maximum) value of $L_w$, assuming $F_w \to \infty$. For example, the values of $L_w(F_w \to \infty)$ which correspond to a damping time of 3.0 ms and to a wiggler field of 1.7 T, 2.0 T, 2.52 T are 224 m, 118 m and 59 m, respectively. Taking into account final value of $F_w$, the length $L_w$ needed to provide $\tau_{x0} = 3.0$ ms for the same values of wiggler field is 146 m, 76 m and 38 m respectively.

For the case $L_w = 152$ m, the horizontal emittance $\gamma \epsilon_{x0}$ given by Eq. (4.11) as a function of $B_w$ and wiggler period $\lambda_w$ is shown in Fig. 4.15. For a fixed value of wiggler period, it takes a minimum at a particular value of the wiggler field that is specified in Fig. 4.16. As one can see from this figure, the optimal field for 76 units of the wiggler magnet ($L_w = 2 \cdot 76 = 152$ m) with period of 10 cm is 1.48 T. It gives the minimum emittance of 125 nm. Using only 26 units of wiggler magnet ($L_w = 2 \cdot 26 = 52$ m) with the same period, the minimum emittance of 201 nm is reached for a wiggler field of 1.7 T.
Figure 4.15: Horizontal emittance $\gamma\epsilon_{x0}$ as a function of wiggler peak field $B_w$ and wiggler period $\lambda_w$ for the case $L_w = 152$ m.

Figure 4.16: Minimum horizontal emittance $\gamma\epsilon_{x0}^{\text{min}}$ as a function of $\lambda_w$ for $L_w = 50$ m, 100 m and 152 m (red curves). On the plot, the values of $\gamma\epsilon_{x0}^{\text{min}}(\lambda_w)$ correspond to the optimal wiggler peak field of $B_w(\lambda_w)$ (blue curves).

It is obvious that large number of high-field wigglers with short period can
improve the horizontal emittance $\gamma \epsilon_0$. At the same time, the longitudinal emittance $\sigma_0 \sigma_\delta m_e c^2 \gamma$ tends slightly to increase, but its value depends on the choice of RF voltage $\hat{V}_{rf}$.

### 4.7.2 Possible wiggler designs and parameters

The number of the FODO cells required for the CLIC damping ring depends on the wiggler parameters. In practice, short period wiggler magnets with $\lambda_w \leq 10$ cm can be manufactured either by permanent magnet technology or by superconducting technology. In contrast, it is technically complicated to fabricate an electromagnetic wiggler magnet with $\lambda_w < 20$ cm and $B_w \sim 2$ T. Moreover, as it will be seen from the next Sec. 4.7.3, wiggler magnets with $\lambda_w > 11$ cm do not produce the CLIC target emittance. However, commonly used permanent magnet technology limits the maximum attainable wiggler peak field at the beam axis to 1.8 T.

A tentative design of a Nd – B – Fe hybrid permanent wiggler magnet with $\lambda_w = 10$ cm, $B_w = 1.7$ T and $L_{ID} = 2$ m is considered for the CLIC damping ring. Engineering feasibility studies were carried out for this wiggler design and its influence on the dynamic aperture was studied. Details are described in Chapter 6.

As an alternative variant, a Nb$_3$Sn superconducting wiggler magnet with parameters in the range of $4$ cm $< \lambda_w < 5$ cm and $2.25$ T $< B_w < 3.05$ T can also be considered, since the fabrication of superconducting wiggles with such parameters is feasible. According to preliminary design of the Nb$_3$Sn wiggler [72], the relation between wiggler period and wiggler peak field on the beam axis is represented in Fig. 4.17.

![Figure 4.17: Relation between wiggler period and wiggler peak field on the beam axis for the Nb$_3$Sn wiggler.](image)

<table>
<thead>
<tr>
<th>$\lambda_w$ (mm)</th>
<th>$I_w$ (kA-t)</th>
<th>$W$</th>
<th>$I$ (kA)</th>
<th>$I/I_c$ (%)</th>
<th>$H_w$ (T)</th>
<th>$H_{cell-max}$ (T)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Nb$_3$Sn</td>
<td>40</td>
<td>130</td>
<td>72</td>
<td>1.80</td>
<td>100</td>
<td>2.25</td>
</tr>
<tr>
<td></td>
<td>40</td>
<td>120</td>
<td>72</td>
<td>1.67</td>
<td>85</td>
<td>2.10</td>
</tr>
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<td></td>
<td>45</td>
<td>125</td>
<td>84</td>
<td>1.50</td>
<td>75</td>
<td>2.52</td>
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<tr>
<td></td>
<td>50</td>
<td>140</td>
<td>84</td>
<td>1.67</td>
<td>85</td>
<td>3.05</td>
</tr>
<tr>
<td>NbTi</td>
<td>50</td>
<td>97</td>
<td>137</td>
<td>0.71</td>
<td>90</td>
<td>2.26</td>
</tr>
</tbody>
</table>

Figure 4.17: Relation between wiggler period and wiggler peak field on the beam axis for the Nb$_3$Sn wiggler.

### 4.7.3 Impact of the IBS effect

Based on the NdBFe and Nb$_3$Sn wiggler technologies mentioned above, we will consider two damping ring designs; the first design with a total wiggler length of $L_w = 152$ m and second one with $L_w = 96$ m.
In the first variant, each straight section includes 18 FODO cells with wigglers, plus 4 wigglers located in the two beta-matching sections. In this case, 76 units of wiggler magnets are used. The circumference of the damping ring is equal to 364.96 m. It is consistent with the integer harmonic number that is equal to 1878. For this reason, the length of the FODO cell (4.6 m) was reduced by 2.73 mm. It induced neglected disturbance of the beam optics but precise rematching was done. Consequently, in the second variant, the ring circumference is 300.48 m ($h = 1878$).

Using the modified Piwinski formalism to compute the effect of IBS, the equilibrium transverse emittances $\gamma\epsilon_x$, $\gamma\epsilon_y$ and equilibrium longitudinal emittance $\epsilon_l = \gamma\sigma_x\sigma_\delta m_e c^2$ as a function of the wiggler peak field $B_w$ and period length $\lambda_w$ were computed in the range of $1.7 \, \text{T} < B_w < 3.0 \, \text{T}$ and $1 \, \text{cm} < \lambda_w < 11 \, \text{cm}$, respectively. The simulations were done for the bunch population of $N_b = 2.56 \times 10^9$ and weak betatron coupling of 0.63 %. The scans for the $L_w = 152$ m case are shown in Fig. 4.18.

As it could be seen from Fig. 4.18a, the horizontal equilibrium emittance has a minimum at particular values of $B_w$ and $\lambda_w$. For a fixed value of wiggler period $\lambda_w$, the horizontal emittance takes the minimum value $\gamma\epsilon_x(\lambda_w)$ at the optimal value of the wiggler field $B_w(\lambda_w)$ that is specified in Fig. 4.19 for both $L_w = 152$ m (solid lines) and $L_w = 96$ m (dashed lines). The red and blue curves correspond to the functions $\gamma\epsilon_x(\lambda_w)$ and $B_w(\lambda_w)$, respectively. As one can see from this figure, the peak field of 1.7 T produced by NdBFe hybrid permanent wiggler magnets with $\lambda_w = 10$ cm is not sufficient to reach the target horizontal emittance of $\gamma\epsilon_x = 450$ nm.

In the CLIC damping rings, the RF frequency is 1875 MHz which is the lowest frequency period length of 1, 6 and 11 cm, from top to bottom.

Figure 4.21 illustrates the dependence of the final horizontal and longitudinal beam emittances, final rms bunch length $\sigma_s$ and the relative energy spread $\sigma_\delta$ on the bunch population. The four curves refer to the two different wiggler designs ($B_w = 2.52 \, \text{T}$ at $\lambda_w = 4.5$ cm (blue curves) and $B_w = 1.7 \, \text{T}$ at $\lambda_w = 10$ cm) (red curves), and two different total lengths of the wigglers $L_w = 152$ m and $L_w = 96$ m (solid and dashed curves, respectively).
Figure 4.18: Transverse equilibrium emittances $\gamma\epsilon_x$ (figure - a), $\gamma\epsilon_y$ (figure - b) and longitudinal emittance $\epsilon_l = \gamma\sigma_x\sigma_t m_e c^2$ (figure - c) as a function of the wiggler field $B_w$ and wiggler period $\lambda_w$ computed with the effect of IBS at the fixed wiggler length $L_w = 152$ m.
Figure 4.19: The minimum horizontal emittance $\gamma\epsilon_x(\lambda_w)$ (red curves) for the optimal value of the wiggler field $B_w(\lambda_w)$ (blue curves) at the fixed value of wiggler period $\lambda_w$. The solid and dashed curves refer to $L_w = 152$ m and $L_w = 96$ m, respectively.

Figure 4.20: The change of RF peak voltage with the wiggler peak field required to maintain the longitudinal emittance near the value of 5000 eVm. The solid and dashed lines correspond to $L_w = 152$ m and $L_w = 96$ m respectively. For a given $L_w$, the three curves refer to wiggler period length of 1, 6 and 11 cm, from top to bottom respectively.
Figure 4.21: The dependence of the final rms bunch length $\sigma_s$, relative energy spread $\sigma_\delta$, horizontal and longitudinal beam emittances on the bunch population. The four curves refer to two different wiggler designs ($B_w = 2.52$ T at $\lambda_w = 4.5$ cm (blue curves) and $B_w = 1.7$ T at $\lambda_w = 10$ cm) (red curves), and to two different total lengths of the wigglers $L_w = 152$ m and $L_w = 96$ m (solid and dashed curves, respectively).

The dependence of the horizontal and longitudinal emittances as well as the relative energy spread and rms bunch length on the RF frequency is shown in Fig 4.22. The six curves refer to the two different wiggler designs ($B_w = 2.52$ T at $\lambda_w = 4.5$ cm & $B_w = 1.7$ T at $\lambda_w = 10$ cm), and two different total lengths of the wigglers ($L_w = 152$ m & $L_w = 96$ m). The dotted lines show the change of the beam qualities due to the increase of RF voltage by +100 kV. Doubling the RF frequency to 3750 MHz increases both transverse emittances by about 16%. At the same time the longitudinal emittance decreases by about 25%. An increase of the RF voltage by 100 kV also results in the increase of both transverse emittances by about 18 nm and the decrease of the longitudinal emittance by about 350 eVm.

Note that both simulations presented in Fig 4.21 and Fig 4.22 were done for a weak betatron coupling of 0.63 %, and assuming zero vertical dispersion invariant $\langle H_y \rangle$. Note also that the values of RF voltage which are indicated near the curves correspond to the longitudinal
emittance of 5000 eVm at the bunch population of $2.56 \times 10^9$ and to the RF frequency of 1875 MHz. The RF voltage in the simulation was not changed with RF frequency or bunch population. The smallest transverse emittances is achieved for the lower RF voltage and frequency.

Figure 4.22: The dependence of the horizontal and longitudinal emittances as well as the relative energy spread and rms bunch length on the RF frequency.

The change of the equilibrium energy spread $\sigma_\delta$ with $L_w$ in the range $96 \leq m \leq L_w$ 152 m is negligible. It depends only on the peak wiggler field $B_w$ and thus is not a significant factor for specifying the total length of the wiggler magnets. Figure 4.23 presents the rms bunch length $\sigma_s$ and rms relative energy spread $\sigma_\delta$ as a function of the wiggler field $B_w$ and wiggler period length $\lambda_w$ at the fixed total wiggler length $L_w = 152$ m. Usually, the equilibrium bunch length should be short to minimize the bunch compression required after the damping ring. However, it is also desirable to keep the bunch length fairly long to reduce the peak current in the ring to reduce the emittance growth due to intrabeam scattering as well as the Touschek scattering rate and to increase the thresholds for longitudinal single-bunch instabilities.
Figure 4.23: RMS bunch length $\sigma_s$ (figure a) and RMS relative energy spread $\sigma_\delta$ (figure b) computed including the effect of IBS as a function of the wiggler field $B_w$ and wiggler period length $\lambda_w$ at the fixed total wiggler length $L_w = 152$ m ($C = 364.96$ m).

Figure 4.24 shows the time evolution of the two horizontal and longitudinal emittances, relative energy spread, and bunch length for different wiggler fields at $L_w = 152$ m. The total time span from the injection to the steady-state beam properties does not depend on the wiggler period.

The momentum compaction is decreased with increasing wiggler field. For the damping ring designs with $L_w = 152$ m and $L_w = 96$ m, the momentum compaction changes from $8.07 \times 10^{-5}$ to $7.63 \times 10^{-5}$ and from $9.94 \times 10^{-5}$ to $9.6 \times 10^{-5}$ if the wiggler field increases from 1.7 T to 3 T.
Figure 4.24: The time evolution of the horizontal and longitudinal emittances, relative energy spread, and bunch length for different wiggler fields at $L_w = 152$ m (C=364.96 m). The red curves correspond to the wiggler peak field of 1.7 T and period length of 10 cm. The blue curves refer to the wiggler period length of 4.5 cm for the different wiggler fields.

### 4.8 Store time and number of the bunch trains

Taking into account that two trains (train pairs) with a gap of 25.6 ns are injected and extracted simultaneously during one machine pulse, the maximum number of train pairs which can be accommodated in the ring with circumference $C$ is defined as

$$N_{2trains}^{max} = \frac{C}{T_{2trains}} = \frac{hc}{f_{rr}T_{2trains}}$$  \hspace{1cm} (4.24)

Here $T_{2trains}$ is the length of the two bunch train plus the gap between them plus the gap between subsequent pairs,

$$T_{2trains} = 2(k_{bt} - 1)\tau_b + 25.6 \text{ ns} + \tau_k$$  \hspace{1cm} (4.25)

where the bunch train has $k_{bt}$ bunches with a bunch spacing of $\tau_b$ and $\tau_k$ is a gap between train pairs to allow a kicker to rise or fall for injection and extraction. The repetition rate is $f_{rr} = 150$ Hz. To keep the beam loading of the cavity in the damping ring almost constant, the train pairs have to be uniformly distributed around the ring.
Taking into account IBS, the time span $T_{eq}$ between the moment of the beam injection ($t = 0$) and the moment when the size of the same beam reaches its equilibrium value is shown in Fig. 4.25 as a function of the wiggler peak field. The simulations were done for the two damping ring designs with $L_w = 152$ m (solid line) and $L_w = 96$ m (dashed line).

![Figure 4.25: The time span $T_{eq}$ between the moment of the beam injection ($t = 0$) and the moment when the size of the same beam reaches its equilibrium value is shown as a function of the wiggler peak field. The solid and dashed line refers to the damping ring design with $L_w = 152$ m and $L_w = 96$ m, respectively.](image)

The minimum number of train pairs which can be accommodated in the ring to provide extraction of equilibrium beam with repetition rate $f_{rr}$ is defined as

$$N_{\text{min}}^{\text{2trains}} = f_{rr} T_{eq}$$  \hspace{1cm} (4.26)

Assuming that the kicker’s rise and fall times do not exceed 25 ns and taking into account that two trains with $k_w = 110$ bunches, $\tau_b = 16$ cm and gap of 25.6 ns are injected and extracted simultaneously during one machine pulse, the possible number of train pairs, which can be stored in the damping ring, is limited by $2 \leq N_{\text{2trains}} \leq 7$ for the $L_w = 152$ m ($C = 364.96$ m) damping ring design with $1.7 \, \text{T} \leq B_w \leq 2.8 \, \text{T}$, and by $2 \leq N_{\text{2trains}} \leq 6$ for the $L_w = 96$ m ($C = 300.48$ m) damping ring design with $2.1 \, \text{T} \leq B_w \leq 3.0 \, \text{T}$, respectively. The nominal store time is 13.3 ms.

### 4.9 Summary

Three variants of the CLIC damping ring design have been considered. The general lattice parameters of these designs are listed in Table 4.8 while the parameters of the extracted beam are listed in Table 4.9.
Table 4.8: General lattice parameters.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Symbol</th>
<th>RING 1</th>
<th>RING 2</th>
<th>RING 3</th>
<th>Unit</th>
</tr>
</thead>
<tbody>
<tr>
<td>Energy</td>
<td>$E$</td>
<td>2.42</td>
<td>2.42</td>
<td>2.42</td>
<td>GeV</td>
</tr>
<tr>
<td>Circumference</td>
<td>$C$</td>
<td>364.96</td>
<td>364.96</td>
<td>300.48</td>
<td>m</td>
</tr>
<tr>
<td>Revolution time</td>
<td>$T_0$</td>
<td>1216.53</td>
<td>1216.53</td>
<td>1001.6</td>
<td>ns</td>
</tr>
<tr>
<td>Total length of wigglers</td>
<td>$L_w$</td>
<td>152</td>
<td>152</td>
<td>96</td>
<td>m</td>
</tr>
<tr>
<td>Number of wigglers</td>
<td>$N_w$</td>
<td>76</td>
<td>76</td>
<td>48</td>
<td></td>
</tr>
<tr>
<td>Length of wiggler</td>
<td>$L_{ID}$</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>m</td>
</tr>
<tr>
<td>Wiggler peak field</td>
<td>$B_w$</td>
<td>1.7</td>
<td>2.52</td>
<td>2.52</td>
<td>T</td>
</tr>
<tr>
<td>Wiggler period length</td>
<td>$\lambda_w$</td>
<td>10</td>
<td>4.5</td>
<td>4.5</td>
<td>cm</td>
</tr>
<tr>
<td>Field of the bending magnet</td>
<td>$B_a$</td>
<td>0.932</td>
<td>0.932</td>
<td>0.932</td>
<td>T</td>
</tr>
<tr>
<td>Bending angle</td>
<td>$\theta$</td>
<td>3.6$^\circ$</td>
<td>3.6$^\circ$</td>
<td>3.6$^\circ$</td>
<td></td>
</tr>
<tr>
<td>Length of the TME cell</td>
<td>$L_{TME}$</td>
<td>1.73</td>
<td>1.73</td>
<td>1.73</td>
<td>m</td>
</tr>
<tr>
<td>Number of the TME cell</td>
<td>$N_{TME}$</td>
<td>96</td>
<td>96</td>
<td>96</td>
<td></td>
</tr>
<tr>
<td>Bending radius</td>
<td>$\rho$</td>
<td>8.67</td>
<td>8.67</td>
<td>8.67</td>
<td>m</td>
</tr>
<tr>
<td>Length of the bending magnet</td>
<td>$L_0$</td>
<td>0.545</td>
<td>0.545</td>
<td>0.545</td>
<td>m</td>
</tr>
<tr>
<td>Energy loss per turn</td>
<td>$U_0$</td>
<td>2.0</td>
<td>3.96</td>
<td>2.63</td>
<td>MeV</td>
</tr>
<tr>
<td>Relative damping factor</td>
<td>$F_w$</td>
<td>4.65</td>
<td>10.22</td>
<td>6.45</td>
<td></td>
</tr>
<tr>
<td>Horizontal damping time</td>
<td>$\tau_x$</td>
<td>2.96</td>
<td>1.49</td>
<td>1.85</td>
<td>ms</td>
</tr>
<tr>
<td>Vertical damping time</td>
<td>$\tau_x$</td>
<td>2.96</td>
<td>1.49</td>
<td>1.85</td>
<td>ms</td>
</tr>
<tr>
<td>Longitudinal damping time</td>
<td>$\tau_p$</td>
<td>1.48</td>
<td>0.745</td>
<td>0.925</td>
<td>ms</td>
</tr>
<tr>
<td>Horizontal tune</td>
<td>$\nu_x$</td>
<td>69.82</td>
<td>69.82</td>
<td>66.18</td>
<td></td>
</tr>
<tr>
<td>Vertical tune</td>
<td>$\nu_y$</td>
<td>33.7</td>
<td>33.7</td>
<td>30.23</td>
<td></td>
</tr>
<tr>
<td>Horizontal natural chromaticity</td>
<td>$\partial \nu_x / \partial \delta$</td>
<td>-105.2</td>
<td>-103.4</td>
<td>-97.0</td>
<td></td>
</tr>
<tr>
<td>Vertical natural chromaticity</td>
<td>$\partial \nu_y / \partial \delta$</td>
<td>-135.0</td>
<td>-139.1</td>
<td>-133.9</td>
<td></td>
</tr>
<tr>
<td>Momentum compaction</td>
<td>$\alpha_p$</td>
<td>0.807</td>
<td>0.782</td>
<td>0.972</td>
<td>$\times 10^{-4}$</td>
</tr>
<tr>
<td>RF frequency</td>
<td>$f_{rf}$</td>
<td>1875</td>
<td>1875</td>
<td>1875</td>
<td>MHz</td>
</tr>
<tr>
<td>RF wave length</td>
<td>$\lambda_{rf}$</td>
<td>0.16</td>
<td>0.16</td>
<td>0.16</td>
<td>m</td>
</tr>
<tr>
<td>RF peak voltage</td>
<td>$V_{rf}$</td>
<td>2250</td>
<td>4225</td>
<td>3030</td>
<td>kV</td>
</tr>
<tr>
<td>Harmonic number</td>
<td>$h_f$</td>
<td>2281</td>
<td>2281</td>
<td>1878</td>
<td></td>
</tr>
</tbody>
</table>

There are only two differences between these designs which are the following: 1) the number of the wiggler FODO cells and 2) the wiggler parameters. Other block-structures such as the arc, wiggler FODO cell, dispersion suppressor, beta-matching section, and injection/extraction region are the same, as described in Sections (4.2), (4.4), (4.5), and (4.6.1). The damping ring layout is a racetrack for all three designs.

The RING 1 design is optimized for the NdFeB permanent magnet wiggler with $\lambda_w = 10$ cm and $B_w = 1.7$ T. The straight sections comprise 76 NdFeB wiggler magnets. The RING 2 design is similar to the RING 1, but superconducting Nb$_3$Sn wigglers are used instead of the NdFeB wigglers. In the RING 3 the same superconducting Nb$_3$Sn wigglers are used but their number is reduced to 48 units, which shortens the circumference of the RING 3 to 300.48 m.

The betatron tunes (working point) for all three designs have been chosen to be sufficiently far from major nonlinear resonances, so as to allow for good dynamic aperture and
Table 4.9: Parameters* of the extracted beam.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Symbol</th>
<th>RING 1</th>
<th>RING 2</th>
<th>RING 3</th>
<th>Unit</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bunch population</td>
<td>$N_{bp}$</td>
<td>2.56</td>
<td>2.56</td>
<td>2.56</td>
<td>$\times 10^9$</td>
</tr>
<tr>
<td>Bunches per train</td>
<td>$k_{bt}$</td>
<td>110</td>
<td>110</td>
<td>110</td>
<td></td>
</tr>
<tr>
<td>Maximum number of bunch trains</td>
<td>$N_{max}^{trains}$</td>
<td>14</td>
<td>14</td>
<td>12</td>
<td></td>
</tr>
<tr>
<td>Minimum number of bunch trains</td>
<td>$N_{min}^{trains}$</td>
<td>4</td>
<td>4</td>
<td>4</td>
<td></td>
</tr>
<tr>
<td>Norm. horizontal emittance w/o IBS</td>
<td>$\gamma \epsilon_{x0}$</td>
<td>131</td>
<td>79</td>
<td>95</td>
<td>nm</td>
</tr>
<tr>
<td>Norm. horizontal emittance with IBS</td>
<td>$\gamma \epsilon_{x}$</td>
<td>540</td>
<td>380</td>
<td>430</td>
<td>nm</td>
</tr>
<tr>
<td>Norm. vertical emittance with IBS</td>
<td>$\gamma \epsilon_{y}$</td>
<td>3.4*</td>
<td>2.4*</td>
<td>2.7*</td>
<td>nm</td>
</tr>
<tr>
<td>Norm. longitudinal emittance** with IBS</td>
<td>$\epsilon_t$</td>
<td>4990</td>
<td>4985</td>
<td>5000</td>
<td>eVm</td>
</tr>
<tr>
<td>RMS bunch length w/o IBS</td>
<td>$\sigma_{s0}$</td>
<td>1.21</td>
<td>1.25</td>
<td>1.21</td>
<td>mm</td>
</tr>
<tr>
<td>RMS energy spread w/o IBS</td>
<td>$\sigma_{50}$</td>
<td>0.0915</td>
<td>0.113</td>
<td>0.111</td>
<td>%</td>
</tr>
<tr>
<td>RMS bunch length with IBS</td>
<td>$\sigma_{s}$</td>
<td>1.65</td>
<td>1.51</td>
<td>1.5</td>
<td>mm</td>
</tr>
<tr>
<td>RMS energy spread with IBS</td>
<td>$\sigma_{5}$</td>
<td>0.125</td>
<td>0.136</td>
<td>0.137</td>
<td>%</td>
</tr>
<tr>
<td>Horizontal IBS growth time</td>
<td>$T_x$</td>
<td>3.89</td>
<td>1.88</td>
<td>2.34</td>
<td>ms</td>
</tr>
<tr>
<td>Longitudinal IBS growth time</td>
<td>$T_p$</td>
<td>5.57</td>
<td>4.403</td>
<td>4.83</td>
<td>ms</td>
</tr>
</tbody>
</table>

* Note that the parameters in this table were computed for the betatron coupling $\epsilon_{y0}/\epsilon_{x0} = 0.0063$ and zero vertical dispersion.

** Note that $\epsilon_t = \gamma \sigma_s \sigma_5 m_0 c^2$.

To stay away from major coupling resonances, which reduces the sensitivity of the vertical emittance to sextupole misalignment and quadrupole rotation errors.

In spite of the fact that the transverse emittances in the RING 1 design are larger than the transverse emittances in the RING 2 and RING 3 designs, the damping ring design RING 1 with the NdFeB permanent magnet wigglers is studied in the next chapters because a concrete design for the NdFeB permanent wiggler with $\lambda_w = 10$ cm and $B_w = 1.7$ T was developed while writing this thesis. In particular, the field map for this wiggler was known, which allowed detailed studies of the a nonlinear wiggler effect on the dynamic aperture. A tentative design of the superconducting Nb$_3$Sn wiggler was suggested only recently. For this reason, the superconducting wiggler scenarios were not studied in detail in the framework of the present thesis.

In the following, we will, therefore, consider the damping ring design RING 1. Layout of this CLIC damping ring is shown in Fig. 4.26.
Figure 4.26: Layout of the CLIC damping ring.
Chapter 5

Non-linear optimization of the CLIC damping ring lattice

5.1 Chromaticity

Particles with different momentum gain different focusing strength in the quadrupoles and, as a consequence, have different betatron oscillation frequency. The chromaticity is defined as the variation of the betatron tunes $\nu_x$ and $\nu_y$ with the relative momentum deviation $\delta = \Delta p/p$:

$$\xi_x = \frac{\partial \nu_x}{\partial \delta}, \quad \xi_y = \frac{\partial \nu_y}{\partial \delta}$$

Sometimes the relative chromaticity is defined as $\xi/\nu$. A big value of chromaticity implies that the beam will occupy a fairly large area in the tune diagram. Therefore, many resonances will be excited and affect the beam stability. For example, horizontal and vertical natural chromaticities of the CLIC damping ring (design RING 1) are $\xi_x = -105.2$ and $\xi_y = -135$ respectively. Thus, tune shift $\Delta \nu$ due to the momentum deviation $\Delta p/p = \pm 0.5\%$ of injected beam will exceed $\Delta \nu > \pm 1$, which is unacceptable. Moreover, in the case of a bunched beam the chromaticity causes a transverse instability called ”head-tail” effect. The wake field produced by the head of the bunch excites an oscillation of the tail of the bunch. The growth rate of this instability is much faster for negative than for positive chromaticity values and vanishes for zero chromaticity. Therefore, most storage rings operate with zero or slightly positive chromaticity.

5.1.1 Natural chromaticity

The chromaticity produced only by the elements of the linear lattice such as quadrupoles and dipoles is called ”natural” chromaticity. Horizontal and vertical natural chromaticities of strong focusing ring are always negative. To compensate the natural chromaticity, nonlinear elements such as sextupole magnets have to be introduced into the lattice. Using only first
order terms in the momentum expansion

\[
\frac{e}{p} = \frac{e}{p_0(1 + \delta)} \approx \frac{e}{p_0}(1 - \delta) + \mathcal{O}(\delta^2)
\]

where \( \delta = \Delta p/p \), the natural chromaticity of a general combined-function magnet is given by [73]

\[
\frac{\partial \nu_x}{\partial \delta} = -\frac{1}{4\pi} \int_0^L \left[ \beta(k + 2h^2 - 2hkD + h'D') - \beta hD(h^2 + k) - \gamma hD \right] ds
\]

\[
\frac{\partial \nu_y}{\partial \delta} = \frac{1}{4\pi} \int_0^L \left[ \beta(k - hkD + h'D') + \gamma hD \right] ds
\]

(5.1)

Here, \( h(s) = \frac{e}{p_0} B_y = \frac{1}{\rho(s)} \). The \( h'(s) \) is determined by the dipole fringing field, and \( k \) is the field gradient \( \left( \partial B_y/\partial x \right)/(B\rho) \).

Chromaticity caused by the quadrupole is a particular case of Eq. (5.1) obtained when \( h(s) = h'(s) = 0 \). Thus, the horizontal \( (\xi_x) \) and vertical chromaticity \( (\xi_y) \) produced by quadrupoles are defined as

\[
\xi_x = \frac{\partial \nu_x}{\partial \delta} = \frac{1}{4\pi} \int_{s_0}^{s_0+C} \beta_x(s) K_1(s) ds
\]

\[
\xi_y = \frac{\partial \nu_y}{\partial \delta} = \frac{1}{4\pi} \int_{s_0}^{s_0+C} \beta_y(s) K_1(s) ds
\]

(5.2)

where \( K_1 \equiv k = \frac{e}{p_0} \frac{\partial B_x}{\partial x} \). For the CLIC damping ring, the chromaticity produced by the pure bending magnets, which do not have any gradient field, is negligible compared with the chromaticity due to the quadrupoles.

Figure 5.1: The horizontal \( (\xi_x) \) and vertical natural chromaticity \( (\xi_y) \) along the CLIC damping ring.
The quadrupole gradient $k$ is positive ($k > 0$) if the quadrupole provides focusing in the horizontal plane and it is negative ($k < 0$) if the quadrupole provides defocusing in the same plane. The $\beta_x$ takes maximum and minimum values in the focusing and defocusing quadrupoles, respectively. The $\beta_y$ takes minimum and maximum values in the focusing and defocusing quadrupoles, respectively. This is the reason why in the strong focusing rings, $\xi_x$ and $\xi_y$ are always negative. The natural chromaticities $\xi_x$ and $\xi_y$ along the CLIC damping ring are shown in Fig. 5.1.

5.1.2 Chromaticity contribution from sextupole magnets

Using sextupoles allows to correct the chromaticity because for off-momentum particles the closed orbit is displaced with respect to the reference orbit by a quantity $D\delta$. Passing the sextupole, an off-momentum particle with initial coordinate $(x + D\delta, y)$ receives a kick

\[
\begin{align*}
x' &= - \left[ D\delta x + \frac{1}{2}(D\delta)^2 + \frac{1}{2}(x^2 - y^2) \right] K_2 l \\
y' &= [D\delta y + xy] K_2 l
\end{align*}
\]

(5.3)

where $K_2 = \frac{e}{\rho_0} \frac{\partial^2 B_y}{\partial x^2}$ is the normalized sextupole strength. The first-order contribution to the chromaticity is given by

\[
\begin{align*}
\frac{\partial \nu_x}{\partial \delta} &= \frac{1}{4\pi} \int_{s_0}^{s_0 + C} \beta_x(s) K_2(s) D(s) ds \\
\frac{\partial \nu_y}{\partial \delta} &= -\frac{1}{4\pi} \int_{s_0}^{s_0 + C} \beta_y(s) K_2(s) D(s) ds
\end{align*}
\]

(5.4)

The most efficient compensation is to correct the natural chromaticity locally, that means to insert sextupoles at each quadrupole. For the damping ring a localized correction is not possible since, it comprises two dispersion-free long straight sections. In this case, the natural chromaticity produced in the straight sections have to be corrected only by sextupoles inserted in the arcs.

As one can see from Eq. (5.4) and Eq. (5.2):

- To perform an efficient correction of the $\xi_x$, the sextupoles with $K_2 > 0$ have to be inserted in the places where the $\beta_x$ functions have a high value and $\beta_y \ll \beta_x$.

- To correct $\xi_y$ efficiently, the sextupoles with $K_2 < 0$ have to be inserted at positions where the $\beta_y$ functions have high values and $\beta_x \ll \beta_y$.

- To minimize the sextupole strengths, it is important to place them at positions where $D_x$ is as high as possible and the betatron functions have a good split.

These are common principles of straightforward chromaticity correction. However, the sextupoles introduce harmful effects due to the additional nonlinearities which are defined by the other two terms in Eq. (5.3)

- the second-order chromatic aberration related with $(D\delta)^2$

- the geometrical aberrations (geometrical terms) proportional to $(x^2 - y^2)$ and $xy$. 

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The challenge is that a small target emittance (small dispersion, high phase advance per cell) entails a large natural chromaticity. The straightforward correction of which by the chromatic sextupoles induces strong nonlinearities (nonlinear deflections, ”kicks”) which consequently limit the dynamic aperture. In order to maximize the dynamic aperture, the nonlinearities must be minimized by choosing carefully the phase advance between sextupoles or by adding additional so-called harmonic sextupoles. Compensation of both natural chromaticity and of the nonlinearities by a proper arrangement of the sextupoles along the CLIC damping ring is studied in the next section.

5.2 Nonlinear particle dynamics

The analysis of the nonlinearities starts by defining the nonlinear Hamiltonian for single particle motion, \( H(x, p_x, y, p_y, \delta; s) \). A perturbation approach provides a very useful insight into the nonlinearities and their effects such as the strengths of specific resonances. It allows formulating the basic principles of sextupole arrangement to cancel or at least minimize the nonlinearities. Then numerical tools based on particle tracking are used to find the sextupole strengths for a particular lattice.

5.2.1 Linear dynamic

In the ultrarelativistic limit, the general Hamiltonian for a charged particle of mass \( m \) and charge \( e \) in a magnetic vector potential \( \mathbf{A} \) is given by

\[
H = -\sqrt{(1 + \delta)^2 - (p_x - \frac{e}{p_0} A_x)^2 - (p_y - \frac{e}{p_0} A_y)^2 - \frac{e}{p_0} A_z}
\]

where \( p_0 = m_0 c \) and \( \delta \equiv (p - p_0)/p_0 \) are the design momentum and the momentum deviation, respectively. Ignoring fringe fields, the multipole expansion of the vector potential \( \mathbf{A} \) can be written as

\[
\frac{e}{p_0} A_x(s) = 0, \quad \frac{e}{p_0} A_y(s) = 0
\]

\[
\frac{e}{p_0} A_s(s) \equiv -\text{Re} \sum_{n=1}^{\infty} \frac{1}{n!} (b_n(s) + ia_n(s))(x + iy)^n
\]

(5.5)

Where \( b_n \) and \( a_n \) are the normal and skew field components respectively. The normal field components are defined as

\[
b_n = \frac{1}{B \rho (n-1)!} \left. \frac{\partial^{n-1} B_y(x, y)}{\partial x^{n-1}} \right|_{y=0} = \frac{K_{n-1}}{(n-1)!}
\]

(5.6)

Applying the adiabatic approximation by taking advantage of the fact that the synchrotron oscillations are in general much slower than the betatron oscillations, the moment deviation can be viewed as a slowly varying parameter (rather than a dynamic variable) so that the longitudinal motion decouples from the transverse one. For simplicity, we assume that the lattice is modelled by a piece-wise constant field consisting of dipole, quadrupoles and sextupoles. One should select only terms \( b_n \) if the 2n – pole magnet is in normal orientation and only terms \( a_n \) if the magnet is skew. For simplification, let us assume that only normal components of the field are present in the ring, as is the case for the nominal optics.
From the perturbation theory point of view, the Hamiltonian can be divided into two parts:

\[ H \equiv H_0 + V \]

where \( H_0 \) is the linear part of Hamiltonian for which the equation of motion can be solved exactly and \( V \) is a Hamiltonian perturbation. Considering a ring, that consists of dipole and quadrupole magnets only, the \( H_0 \) for the linear betatron motion of an on–momentum particle is expressed by

\[
H_0 = \frac{p_x^2 + p_y^2}{2} + \frac{b_2(s)}{2}(x^2 - y^2) \]  

(5.7)

The solution is found as:

\[
x = \sqrt{2J_x}\beta_x(s) \cos \Psi_x(s) \\
p_x = -\sqrt{2J_x}\frac{1}{\beta_x(s)}[\sin \Psi_x(s) + \alpha_x(s) \cos \Psi_x(s)]
\]  

(5.8)

where

\[
\Psi_x(s) = \int_{s_0}^{s} \frac{ds'}{\beta_x(s')} + \phi_x(s_0)
\]

Here, \( \alpha_x = -\beta'_x/2 \). The same holds for the vertical plane, where one should replace the subscript \( x \) by \( y \) in Eq. (5.8). This solution describes an ellipse in the phase space with area of \( E = 2\pi J \). Moreover, \( J_x \) is constant. Eq. (5.8) can be inverted for \( J_x \) and \( \phi_x \) which results in

\[
\Psi_{0x}(p_x, x, s) = -\arctan \left[ \beta_x(s) \frac{p_x}{x} + \alpha_x \right] \\
J_{0x}(p_x, x, s) = \frac{1}{2\beta_x(s)}\left\{ x^2 + [\beta_x(s)p_x + \alpha_x(s)x]^2 \right\}
\]  

(5.9)

Note, that \( 2J_x \) is the well-known Courant-Snyder invariant since \( p_x = x' \) for the linear motion. The coordinates \( \{p_x, x, p_y, y\} \) are conjugate according to the rules of Hamilton but neither term is a constant of motion. To represent the linear terms using a constant of motion, it is needed to make the canonical transformation of \( \{p_x, x, p_y, y\} \) to the new coordinates, the action \( J \) and the angle \( \Psi \). In this case, the Hamiltonian becomes cyclic in \( \Psi \). For the linear motion, the action \( J_{0x} \) is constant. The perturbed motion is obtained by

\[
J'_x = -\frac{\partial V}{\partial \Psi_x}, \quad \Psi'_x = \frac{\partial V}{\partial J_x} \\
J'_y = -\frac{\partial V}{\partial \Psi_y}, \quad \Psi'_y = \frac{\partial V}{\partial J_y}
\]  

(5.10)

### 5.2.2 Perturbation theory for multipole expansion of Hamiltonian

Sextupole, octupole and other high order multipole fields, whose vector potential is described by Eq. (5.5), all add a nonlinear part to the linear Hamiltonian \( H_0 \). For two degrees of freedom, the total Hamiltonian can be represented as a multipole expansion.

\[
H(s) = H_0 + \sum_{m_x, m_y} V_{m_x, m_y}(s)x^{m_x}y^{m_y}
\]  

(5.11)
where \( m_x \) and \( m_y \) are positive integers. They start from the sextupole where \( m_x + m_y = 3 \). The expansion coefficients \( V_{m_x, m_y} \) are derived from Eq. (5.5) for vector potential as

\[
V_{m_x, m_y} = \begin{cases} 
    b_{m_x + m_y} \frac{(m_x + m_y - 1)!}{m_x! m_y!} (i)^{m_y} & \text{if } m_y \text{ even} \\
    a_{m_x + m_y} \frac{(m_x + m_y - 1)!}{m_x! m_y!} (i)^{m_y+1} & \text{if } m_y \text{ odd}
\end{cases}
\]

where \( b_{m_x + m_y} \) and \( a_{m_x + m_y} \) are the normal and skew field coefficients given by Eq. (5.6).

We assume that there are no skew components of the field in the ring. Thus, \( m_y \) will be always even in our consideration. The nonlinear terms are distributed in azimuthal position \( s \) around the ring. It is necessary to find a canonical transformation which transforms the system \( \{ H, \varepsilon, \phi \} \) into a new system \( \{ K, J, \varphi \} \) where the Hamiltonian depends on the action variable \( J \) only. Following the classical perturbation theory, we should choose a generation function \( F \) which is mixed in old and new canonical variables

\[
F(J, \phi, \theta) = J\phi + S(J, \phi, \theta)
\]

The transformation equations derived from the above equation are expressed

\[
\begin{align*}
\varepsilon &= J + \partial S(J, \phi, \theta) / \partial \phi \\
\varphi &= \phi + \partial S(J, \phi, \theta) / \partial J \\
K &= H + \partial S(J, \phi, \theta) / \partial \theta
\end{align*}
\]

(5.13)

Such technique can be found in \([74, 75, 76]\). Here we just give results since they will be used in the following sections. The transformation which removes the "time" dependence (in other words "s" dependence) from \( H \) is found as

\[
F(J, \phi, \theta) = J_x \phi_x + J_y \phi_y + \sum_{p=-\infty}^{\infty} \sum_{jklm} J_{x}^{j+k} J_{y}^{l+m} h_{jklm} e^{i[(j-k)(\phi_x + \nu_x \theta) + (l-m)(\phi_y + \nu_y \theta) + p\theta + \phi_{jklm}(p)]}
\]

(5.14)

where the Fourier components \( h_{jklm} \) are defined as

\[
h_{jklm} e^{i\phi_{jklm}(p)} \propto \frac{1}{2\pi} \int_{0}^{2\pi} V_{m_x, m_y} \beta_x^{j+k} \beta_y^{l+m} e^{i[(j-k)(\phi_x - \nu_x \theta) + (l-m)(\phi_y - \nu_y \theta) - p\theta]} d\theta
\]

(5.15)

and

\[
\begin{align*}
m_x &= j + k \\
m_y &= l + m \\
n_x &= j - k \\
n_y &= l - m \\
p &= \text{integer}
\end{align*}
\]

(5.16)

Here \( \theta = \frac{2\pi (s - s_0)}{L} \), where \( L \) is the ring circumference, \( \nu_{x,y} \) is the familiar betatron wavenumber along the ring tune and \( p \) is the harmonic of the perturbation driving the resonance. If \( n_x \)
and \( n_y \) have the same sign the resonance is called a sum resonance otherwise it is called a difference resonance. If the betatron working point \((\nu_x, \nu_y)\) is close to the single resonance
\[ n_x \nu_x + n_y \nu_y = p \]
then the perturbation will be dominated by this resonance and the other terms in Eq. (5.14) may be neglected. The working point \((\nu_x, \nu_y)\) has to be chosen to stay away from the low-order resonance lines defined by
\[ \nu_x = p_1, \ 3\nu_x = p_2, \ \nu_x + 2\nu_y = p_3, \ \nu_x - 2\nu_y = p_4 \]
\( \{p_1, p_2, p_3, p_4\} \) are integer numbers) in the tune diagram. The final form of the generating function is obtained by carrying out the sum over the Fourier series and over the \( s \) variable for many turns. After averaging, the generating function \( F \) for two degrees of freedom corresponding to the transverse motion of on-momentum particle is written as
\[
F(J, \phi, s) = J_x \phi_x + J_y \phi_y + \sum_{p=-\infty}^{p=\infty} \sum_{jklm} \frac{J_x^{j+k} J_y^{l+m} h_{jklm} \sin[n_x \phi_x + n_y \phi_y + \phi_{jklm}(p)]}{\sin \pi(n_x \nu_x + n_y \nu_y - p)}
\] (5.18)

From Eq. (5.13) the amplitude dependent betatron tune shift with amplitude can be found from
\[
\varphi_{x,y} = \phi_{x,y} + \Delta \phi_{x,y} = \partial F(J_x, J_y, \phi_x, \phi_y, s)/\partial J_{x,y}
\] (5.19)

### 5.2.3 The perturbation depending on \( \delta \)

The strength of the multipole components affecting the particles depends on the particle momentum. A particle with momentum deviation \( \delta \) experiences the strength defined by
\[
b_n(\delta) = \frac{b_n}{1+\delta} = b_n(1 - \delta + \delta^2 - \delta^3 + \ldots)
\] (5.20)

Taking into account the momentum deviation, variables \( x \) and \( y \) transform to [76]
\[
x = \sqrt{\frac{2 J_x \beta_x(s)}{(1+\delta)}} \cos \phi_x(s) + D_x^{(0)} \delta, \quad y = \sqrt{\frac{2 J_y \beta_y(s)}{(1+\delta)}} \cos \phi_y(s)
\] (5.21)

where \( D_x^{(0)} \) denotes the first order horizontal dispersion which is a solution of the equation
\[
D''_x + \left[ \frac{1}{\rho^2(s)} - K_1 \right] D_x = \frac{1}{\rho(s)}
\]

Substituting Eqs. (5.21–5.20) into Eq. (5.11) for multipole Hamiltonian expansion, the absolute value of the Fourier coefficient \( h_{jklm} \) which determines the strength of the resonance can be written in the form
\[
h_{jklm}(g) \propto \frac{1}{2\pi} \int_0^{2\pi} \left[ 2b_3(\theta) |D_x^{(0)}| \delta - b_2(\theta) \right] \frac{J_x^{j+k} J_y^{l+m} \beta_x^{j+k} \beta_y^{l+m}}{\rho^2(s)} d\theta
\] (5.22)

Here, \( g = 1 \) or 0. Only for the chromatic modes \( (g = 1) \) the quadrupole contribution has to be included, otherwise it is equal to zero.
5.2.4 First order chromatic terms and linear chromaticity

According to Eq. (5.22), there are two terms which drive the linear chromaticity and they are independent of the phase variable:

\[
\begin{align*}
h_{11001} & \propto \quad \sum_{i=1}^{\text{quad}} (K_1 l)_i \beta_{x_i} - \sum_{j=1}^{\text{sext}} (K_2 l)_j D_{x_j}^{(0)} \beta_{x_j} \\
h_{00111} & \propto - \quad \sum_{i=1}^{\text{quad}} (K_1 l)_i \beta_{y_i} - \sum_{j=1}^{\text{sext}} (K_2 l)_j D_{x_j}^{(0)} \beta_{y_j}
\end{align*}
\]  

(5.23)

The remaining three terms are given by [77]

\[
\begin{align*}
h_{20001} & = h_{02001}^* \propto \quad \sum_{i=1}^{\text{quad}} (K_1 l)_i \beta_{x_i} e^{i2\mu_{xi}} - \sum_{j=1}^{\text{sext}} (K_2 l)_j D_{x_j}^{(0)} \beta_{x_j} e^{i2\mu_{xj}} \\
h_{00201} & = h_{00021}^* \propto - \quad \sum_{i=1}^{\text{quad}} (K_1 l)_i \beta_{y_i} e^{i2\mu_{yi}} - \sum_{j=1}^{\text{sext}} (K_2 l)_j D_{x_j}^{(0)} \beta_{y_j} e^{i2\mu_{yj}} \\
h_{10002} & = h_{01002}^* \propto \quad \sum_{i=1}^{\text{quad}} (K_1 l)_i D_{x_i}^{(0)} \beta_{x_i}^{1/2} e^{i\mu_{xi}} - \sum_{j=1}^{\text{sext}} (K_2 l)_j \left[ D_{x_j}^{(0)} \right]^2 \beta_{x_j}^{1/2} e^{i\mu_{xj}}
\end{align*}
\]  

(5.24)

Here \(h_{20001}\) and \(h_{00201}\) drive synchro-betatron resonances and generate momentum dependence of the beta functions that can limit the longitudinal acceptance. Whereas term \(h_{10002}\) drives the second order dispersion.

The linear chromaticity is defined as

\[
\begin{align*}
\xi^{(1)}_x & \equiv \left. \frac{\partial \nu_x}{\partial \delta} \right|_{\delta=0} = -\frac{1}{4\pi} \int_{s}^{s+C} \beta_x(s) \left[ K_1(s) - K_2(s) D_x^{(0)}(s) \right] ds \\
\xi^{(1)}_y & \equiv \left. \frac{\partial \nu_y}{\partial \delta} \right|_{\delta=0} = \frac{1}{4\pi} \int_{s}^{s+C} \beta_y(s) \left[ K_1(s) - K_2(s) D_x^{(0)}(s) \right] ds
\end{align*}
\]  

(5.25)

5.2.5 First order geometric terms

Using the Hamiltonian formalism, we can define five terms which drive third order and integer resonances.

\[
\begin{align*}
h_{21000} & = h_{12000}^* \propto - \sum_{i=1}^{N} (K_2 l)_i \beta_{x_i}^{3/2} e^{i\mu_{x_i}} \quad \Rightarrow \quad \nu_x \quad \text{with} \quad \beta_{x_i}^{3/2} \quad \text{term} \\
h_{30000} & = h_{03000}^* \propto - \sum_{i=1}^{N} (K_2 l)_i \beta_{x_i}^{3/2} e^{i3\mu_{x_i}} \quad \Rightarrow \quad 3\nu_x \\
h_{10110} & = h_{01110}^* \propto \sum_{i=1}^{N} (K_2 l)_i \beta_{x_i}^{1/2} \beta_{y_i} e^{i\mu_{x_i}} \quad \Rightarrow \quad \nu_x \quad \text{with} \quad \beta_{x_i}^{1/2} \beta_{y_i} \quad \text{term} \\
h_{10020} & = h_{01020}^* \propto \sum_{i=1}^{N} (K_2 l)_i \beta_{x_i}^{1/2} \beta_{y_i} e^{i(\mu_{x_i} - 2\mu_{y_i})} \quad \Rightarrow \quad \nu_x - 2\nu_y \\
h_{10200} & = h_{01020}^* \propto \sum_{i=1}^{N} (K_2 l)_i \beta_{x_i}^{1/2} \beta_{y_i} e^{i(\mu_{x_i} + 2\mu_{y_i})} \quad \Rightarrow \quad \nu_x + 2\nu_y
\end{align*}
\]  

(5.26)
These terms drive five different betatron modes with frequencies:
\[ \nu_x, \ 3\nu_x, \ \nu_x - 2\nu_y, \ \nu_x + 2\nu_y \] (5.27)

### 5.2.6 Second order geometric terms

The second order modes appear due to cross terms of the first order modes. The terms which are independent of the angle variables drive amplitude dependent tune shift. These effects may be viewed as originating from an amplitude-dependent shift of the closed orbit in the sextupoles. The contribution of these terms to the perturbing Hamiltonian can be expressed as [77]

\[ \Delta H \sim -\frac{1}{64}(3h_{21000}h_{12000} + h_{30000}h_{03000})(2J_x)^2 + \frac{1}{16}(2h_{21000}h_{01110} + h_{10020}h_{01200} + h_{10200}h_{01020})(2J_x)(2J_y) - \frac{1}{64}(4h_{10110}h_{01110} + h_{10020}h_{01200} + h_{10200}h_{01020})(2J_y)^2 \] (5.28)

The remaining terms

\[ \Delta H \sim \frac{1}{64}\left[ 2(h_{30000}h_{12000})_{2\nu_x} + (h_{30000}h_{21000})_{4\nu_x} \right](2J_x)^2 + \frac{1}{64}\left[ 2(h_{30000}h_{01110} + h_{21000}h_{10110} + 2h_{10200}h_{10020})_{2\nu_x} + 2(h_{10200}h_{12000} + h_{21000}h_{01200} + 2h_{10200}h_{01110} + 2h_{10110}h_{01200})_{2\nu_y} + (h_{21000}h_{10020} + h_{30000}h_{01020} + 4h_{10110}h_{10020})_{2\nu_x - 2\nu_y} + (h_{30000}h_{01200} + h_{10200}h_{21000} + 4h_{10110}h_{10200})_{2\nu_x + 2\nu_y} \right](2J_x)(2J_y) + \frac{1}{64}\left[ 2(h_{10200}h_{01110} + h_{10110}h_{01200})_{2\nu_y} + (h_{10200}h_{01200})_{4\nu_y} \right](2J_y)^2 \] (5.29)

drive 8 different betatron modes with the frequencies:
\[ 2\nu_x, \ 4\nu_x, \ 2\nu_y, \ 4\nu_y, \ 2\nu_x - 2\nu_y, \ 2\nu_x + 2\nu_y \] (5.30)

### 5.3 Second order achromat

#### 5.3.1 Conditions for the second order achromat

The second order achromat is an optical system including sextupoles. The second order achromat consists of four or more identical cells constituting the optical system with overall phase advance that is equal to multiple of $2\pi$ in both transverse plane. Further, we will call the second order achromat just the achromat for simplicity.

In order for the geometric aberrations\(^2\) to vanish, the derivatives of the generating function $S$ must be equal to zero at the end of an achromat. The derivatives of $S$ with respect

---

\(^2\)The second and higher order geometric aberrations will be referred to as the second and higher order coefficients, respectively, of the Taylor expansion of the solution of the equations of motion which only depend on the reference momentum $p_0$. In other words, the second order geometric aberrations are defined by the second order matrix elements $T_{ijk}$ where $i, k, j = \{1, 2, 3, 4\}$. Any elements $T_{ijk}$ where one subscript is equal 6 (dependence on $\delta$) will be referred to as second order chromatic aberrations.
to the canonical variables can be written as sums of terms which are linearly independent. Considering a typical term of those sums the following conditions should be satisfied:

$$\int_0^L U_{nx,ny}(s)e^{i(n_x\phi_x+n_y\phi_y)}ds' = 0 \quad (5.31)$$

where $U_{nx,ny}$ are the complex amplitude obtained from Eq. (5.14–5.15), and $L$ is the length of the achromat.

Let us consider an achromat built from of $N$ identical cells of length $l$ and with tunes per cell $\nu^c_x$ and $\nu^c_y$. For any second order achromat the overall tunes must be $N\nu^c_x = \text{integer}$, $N\nu^c_y = \text{integer}$. Assume that the strength of sextupoles have been found such that chromaticities $\xi_x$, $\xi_y$ from Eq. (5.25) are equal to zero. The $U_{nx,ny}$ is a periodic function with period $l$ because all $N$ cells are identical. The equation (5.31) can be written as

$$\int_0^L U_{nx,ny}(s')e^{i(n_x\phi_x+n_y\phi_y)}ds' = \int_0^l U_{nx,ny}(s')e^{i(n_x\phi_x+n_y\phi_y)}\frac{1-e^{2\pi i N(n_x\nu^c_x+n_y\nu^c_y)}}{1-e^{2\pi i(n_x\nu^c_x+n_y\nu^c_y)}}ds' = 0 \quad (5.32)$$

since the sum over $N$ cell can be evaluated by using

$$\sum_{n=0}^N e^{i(nuv)} = e^{iu} \sum_{n=0}^N e^{inv} = e^{iu} \frac{1-e^{iNv}}{1-e^{iv}}$$

To satisfy Eq. (5.32), the unperturbed tunes $\nu^c_x$ and $\nu^c_y$ must avoid the following resonance values:

$$n_x\nu^c_x + n_y\nu^c_y \neq \text{integer} \quad (5.33)$$

and they must satisfy the condition

$$N\nu^c_x = \text{integer}, \quad N\nu^c_y = \text{integer} \quad (5.34)$$

For the achromat with phase advance $2\pi$ which consists of four or more identical cells, the conditions of Eq. (5.33) are reduced to only one requirement $3\nu^c_x, 3\nu^c_y \neq \text{integer}$. In this particular case, the conditions for the second order achromat were originally formulated by K.Brown [78]. "If one combines four or more identical cells consisting of dipole, quadrupole, and sextupole components, with the parameters chosen so that the overall first-order transfer matrix is equal to unity ($+I$) in both transverse planes, then it follows that such a system will have vanishing second-order geometric (on momentum) aberrations”.

Moreover, K.Brown also showed that if the strengths of the sextupoles are adjusted so that one of the second-order chromatic terms $T_{ij6}$ or $T_{2ij6}$ and one of $T_{3ij6}$ or $T_{4ij6}$ are equal to zero then all the second-order chromatic terms except $T_{566}$ become simultaneously zero.

### 5.3.2 $-I$ Principle

Two sextupoles of equal strength, which are placed at the entrance and exit of a minus unity ($-I$) first-order transfer matrix (see Eq. 2.11) in both the x and y transverse planes, will not introduce second-order geometric aberrations outside this transfer matrix. In addition, two equal sextupoles separated by $-I$ do not introduce any second-order dispersion but the beta-beat will be excited (see Appendix B).

Applying the $-I$ principle to arrange the sextupole families for the chromaticity correction, we can be sure that
1. Any sextupole family where adjacent sextupoles are separated by $-I$ transformer will not introduce second-order geometric aberrations.

2. The interlacing of two or more sextupole families, each of which satisfies criterion 1., does not introduce second-order geometric aberrations.

3. Interlacing of one sextupole family with another sextupole family will introduce third and higher-order aberrations.

These statements were also originally formulated by Karl L. Brown [78]. The same compensation would be achieved for two octupoles separated by $-I$, if their strengths are chosen equal but with opposite signs.

The ideal situation is to assemble enough $-I$ transformers so that the different sextupole pairs placed $-I$ apart do not interfere with each other, but this condition is often impossible to achieve.

In our case, non-interlaced $-I$ transformers with thin sextupoles are impossible to realize, because there is not enough space available to arrange them. The small beta and dispersion functions require a sufficient number of strong sextupoles in order to correct the large values of horizontal and vertical chromaticity.

5.4 Sextupole application for the CLIC damping ring: nonlinear optimization

As any modern high performance machine, the CLIC damping ring has a lattice with very strong focusing to meet the requirements for the ultra-low target beam emittance. Moreover, to reduce the extremely strong effect of intra-beam scattering resulting from the ultra-low target emittance, the arcs were designed to provide small betatron and dispersion functions and two long wiggler straight sections which enhance radiation damping were included. As a consequence, to compensate the large natural chromaticity with small optical functions, the strength of sextupoles located in the arcs becomes very strong.

In fact, there are no longer distinct sequential steps between linear and nonlinear lattice optimization, but an iteration between the two becomes necessary. Nonlinear optimization of the damping ring lattice can have a strong impact on the linear optics design. Therefore, at the stage of the linear design, we have provided the possibility to arrange the second order achromats and sextupole families with $-I$ separation between sextupoles. Such flexibility enables us to perform a nonlinear optimization which means

- to determine the necessary number of sextupole families,
- to find their strengths in order to cancel strongest nonlinearities,
- to add, if needed, families of the harmonic sextupoles which can be placed in the dispersion-free regions.

By particle tracking we control the dynamic aperture which represents the indicator of the effectiveness of the nonlinear optimization.
5.4.1 Numerical tools

Commonly used codes include MAD [79], BETA-LNS [80], OPA [81] and RACETRACK [82]. Many of these codes, which originate from the early period of light source design, have been enhanced in an evolutionary way so as to incorporate additional features required in later periods. For example, the more rigorous inclusion of nonlinear lattice studies (BETA-LNS), the more sophisticated inclusion of insertion device effects (RACETRACK and BETA-LNS) or the inclusion of the output from modern one turn map analysis (MAD). There are also Lie algebra based codes designed specifically to produce the coefficients of the one turn map, one of the earliest and most widely used being MARYLIE [83], which provides as output the nonlinear terms in the generator. For the nonlinear optimization we used mainly BETA-LNS and MAD. BETA-LNS code contains an explicit algorithm for minimization of the geometric aberrations which are interpreted in the same way as in Eq. (5.26).

5.4.2 A sextupole scheme for the TME structure

The first order chromaticity correction can be done by using at least two families of sextupoles in the arc. Two possible options for the placement of the sextupoles in the TME cell are shown schematically in Fig. 5.2. The sextupoles (SF,SD) and quadrupoles (QF,QD) correspond to the blue and green rectangles respectively. Option B provides little better split of beta functions at the sextupoles than option A, yielding a slight reduction of sextupole strength. However, the tune shifts with amplitude $\partial \nu / \partial J_{x,y}$ are nearly an order of magnitude larger with option B. The sextupoles should also be placed where the linear optics functions have a weak $\delta$ dependence.

As it was seen in the previous sections, there are two different approaches to group sextupoles around the ring in order to compensate the natural chromaticity and to cancel the first order geometric aberrations.

Figure 5.2: Two options for the sextupole locations in the TME cell of the CLIC damping ring

The first described in Sec. 5.3.2 is to group the sextupoles in pairs separated by the $-I$ linear transfer matrix. By such overall arrangement, one may with two independent families of sextupoles cancel the first order chromaticities Eq. (5.25) driven by $h_{11001}$ and $h_{00111}$ Eq. (5.23) and all first order geometric modes Eq. (5.26). However, this pattern may systematically excite the chromatic modes $h_{20001}$ and $h_{00201}$ Eq. (5.24) which drive the off-momentum beta-beat (see Eqs. B.6–B.7 in Appendix B). The values of $h_{20001}$ and $h_{00201}$ can become comparable to the $h_{11001}$ and $h_{00111}$ that may consequently generate a substantial amount of second order chromaticity Eq. (B.4). This scheme for the chromatic sextupoles can be applied to the damping ring only by interleaving sextupole pairs since the wide separation of the sextupoles for a non-interleaved $-I$ arrangement would make their strength very strong, enhancing the second order effects. However, with interleaved sextupole pairs, we need to control the cross talk between the sextupoles, i.e. the terms of high order.
The second approach described in Sec. 5.3.1 is to design the second order achromat from four or more identical (unit) cells and adjust its betatron phase advance to be a multiple of \(2\pi\) for the horizontal and vertical plane. The linear chromaticity and all the first order chromatic as well as geometric modes are cancelled at the end of the structure. This approach is applied to the CLIC damping ring lattice. The sextupolar achromat is a smart solution for the problem of interleaved sextupole pairs. The achromat condition, Eqs. (5.33–5.34), can be represented in the tune diagram as shown in Fig. 5.3. The blue and green lines correspond to systematic resonances of 3rd order given by Eq. (5.27) and to octupole-like resonances of 4th order given by Eq. (5.30), respectively. If the phase advance of the unit cell \(\{\nu_x^c, \nu_y^c\}\) is on a resonance line (or very close to it) than the strength of the corresponding resonance is strongly amplified. Fourier harmonics \(h(s)_{jkmn}e^{i\phi_{jkmn}(p)}\) produced by each sextupole in the achromat can be represented geometrically as a vector in the complex plane. The integrals over the lattice of achromat become the vector sums of all the complex vectors contributing to the same geometric aberration. According to Eq. (5.29), the octupole-like geometric aberrations of second order due to the cross-talk of sextupoles can be represented as a composition of complex vectors

\[
\sum_j \sum_k (K_2 l_j)(K_2 l_k) F(\beta_j, \beta_k)e^{i2\pi f(\nu_x(j-k)+\nu_y(j-k))}
\]

(5.35)

The \(F(\beta_n, \beta_m)\) is a product of some power of the two \(\beta(s)\) functions. Therefore, if the horizontal and vertical phase advances per one unit cell are multiple of

\[
2\pi \cdot \frac{(2n + 1)}{4}
\]

(5.36)

where \(n\) is an integer number, than the structure consisting of two unit cells will cancel the resonances \(2\nu_x, 2\nu_y, 2\nu_x - 2\nu_y, 2\nu_x + 2\nu_y\), i.e., the octupole-like resonances, because the double betatron frequency becomes \(\pi\). This implies a cancellation of the complex vectors which correspond to the following harmonics: \(\{h_{31000}, h_{20110}\}, \{h_{00310}, h_{01110}\}, \{h_{20020}\},\)}
In the same way, the structure consisting of four unit cells cancels the $\nu_x$, $3\nu_x$, $\nu_x + 2\nu_y$, $\nu_x - 2\nu_y$ resonances. However, in the achromat where the unit cell is tuned to $(2n+1)/4$ the octupole-like resonances $4\nu_x$ and $4\nu_y$ can be excited due to the cross-talk.

Each arc of the CLIC damping ring consists of 48 TME cells. The requirement for the ultra-low final emittance enforces many short bending magnets with TME conditions. The strong effect of intra-beam scattering (IBS) imposes small beta and dispersion functions in the arcs. Therefore, the compact TME cells with strong focusing were chosen. Strong sextupoles are needed to carry out the chromaticity correction in such a lattice since the average dispersion in the arc is only 8.5 mm and the split of the horizontal and vertical beta functions is small. Taking into account the strong IBS effect, the linear and nonlinear optimization of such lattice is a difficult compromise between the final emittance and the dynamic aperture. For example, reducing horizontal phase advance of the TME cell has two effects; On the one hand, the betatron split at the sextupole locations is slightly improving. In addition, the $\beta_x$ and $D_x$ functions become larger, but the horizontal natural chromaticity of the cell is increasing too. The sextupole strength needed for chromaticity correction is slightly decreased which enlarges the dynamic aperture little bit. On the other hand, the transverse emittances grow since firstly the TME detuning factor is increased and secondly IBS effect becomes stronger due to the increase of average value of $\beta_x$ and $D_x$. Detuning vertical phase advance from the $\pi/2$ in the proposed lattice design of TME cell causes significant growth of natural vertical chromaticity. Including in our simulation the strong effect of IBS as it was described in the Chapters 3 and 4, the 96 TME compact cells with phase advance $\Delta \mu_x^{TME} = 0.5833$, $\Delta \mu_y^{TME} = 0.25$ and 76 wigglers located in the two straight sections provide the transverse emittances $\gamma \epsilon_x = 540 \text{ nm}$ and $\gamma \epsilon_y = 3.4 \text{ nm}$.

![Schematic view of the cancellation between sextupole families in the achromat for a TME phase advance $\Delta \mu_x^{TME} = 0.5833 \times 2\pi$, $\Delta \mu_y^{TME} = 0.25 \times 2\pi$.](image)

Taking into account the above considerations, we cannot decrease $\Delta \mu_x$ to less than 0.5833 or make significant changes of $\pi/2 \pm \Delta \mu_y$ in the TME cell to organize the sextupolar achromat, but the high periodicity of the arc allows for the following variant of achromat configuration:

- phase advances of the TME cell $\Delta \mu_x = 0.5833 \times 2\pi$ and $\Delta \mu_y = 0.25 \times 2\pi$. The achromat unit cell consists of 3 TME cells where sextupoles with different strengths are located.
Repeating the unit cell 4 times, we arrange the achromat including 12 TME cells with overall phase advance $\Delta \mu_{xa} = 7 \times 2\pi$ and $\Delta \mu_{ya} = 3 \times 2\pi$. Nine sextupole families can be used in such achromat configuration [84]. The schematic view of achromat and cancellation between sextupole families is shown in Fig. 5.4. The sextupoles in the first and second unit cells constitute $-I$ transformers with the sextupoles located in the third and fourth unit cells respectively. Consequently, horizontal and vertical $-I$ cancellation between every $N$th and $(N + 2)$th cell occurs within the achromat. The phase advances over the unit cell $\Delta \mu_c^x = 1.75 \times 2\pi$ and $\Delta \mu_c^y = 0.75 \times 2\pi$ meet requirement given in Eq. (5.33).

If the sextupoles are rather strong, as expected in the case of CLIC damping ring, seven second order terms arising from the cross talk of the sextupoles have to be compensated or at least minimized (in the literature this is sometimes called a second-order nonlinear optimization):

- 3 phase independent terms from Eq. (5.28) are the contribution to linear tune shifts with amplitude:
  
  \[
  \frac{\partial \nu_x}{\partial J_x}, \quad \frac{\partial \nu_x}{\partial J_y} = \frac{\partial \nu_y}{\partial J_x}, \quad \frac{\partial \nu_y}{\partial J_y},
  \]

- 2 phase dependent terms from Eq. (5.29) which drive the different modes of octupole-like resonances:
  
  $4\nu_x, \ 4\nu_y,$

- 2 phase independent off-momentum terms from Eq. (5.24) which drive the 2nd order chromaticities:

  \[
  \frac{\partial^2 \nu_x}{\partial \delta^2}, \quad \frac{\partial^2 \nu_y}{\partial \delta^2}
  \]

This optimization is carried out numerically. Producing a good dynamic behaviour requires a delicate balancing/setting of various weights to cancel and minimize the terms which are most relevant to the nonlinear motion under consideration.

The contributions to the natural chromaticity due to the two straight sections and the four dispersion suppressors are $\Delta \xi_x = -24.6$ and $\Delta \xi_y = -21.7$. The natural chromaticity of each TME cell is $\Delta \xi_x = -0.84$ and $\Delta \xi_y = -1.18$. Therefore, the sextupoles of one achromat have to introduce the positive amount of $\Delta \xi_{xa} = 3.07$ and $\Delta \xi_{ya} = 2.7$. Obviously in this case, the chromaticity correction becomes nonlocal that leads to a corresponding degradation of the dynamic aperture. An eventual solution may be to organize in the arc the following achromat scheme: S2-S1-S1-S2 where each achromat S2 and S1 consists of four unit cells but the S1 performs a local correction, while the S2 creates the needed positive chromaticity to compensate the straight sections. This variant was not studied yet. However, we can assume that the strength of sextupoles in the S2 achromat will be too strong. The dynamic aperture may deteriorate to an unacceptable value.

Nine independent sextupoles were installed in one unit cell of the achromat at the positions which correspond to the option A (see Fig. 5.2). Using BETA-LNS code, we optimized their strengths to meet the required chromaticity $\Delta \xi_{xa} = 3.07, \Delta \xi_{ya} = 2.7$ the end of one achromat and to minimize: 3 phase independent constraints $\partial \nu_x/\partial J_x, \partial \nu_x/\partial J_y = \partial \nu_y/\partial J_x, \partial \nu_y/\partial J_y$ and 2 phase independent off-momentum constraints $\partial^2 \nu_x/\partial \delta^2, \partial^2 \nu_y/\partial \delta^2$. 

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Figure 5.5: Fourier harmonics in the case when the damping ring consists of two arcs only: 
A the spectrum \( h(p) \) of the first order geometric aberrations \( h_{21000}, h_{10110}, h_{30000}, h_{10200}, h_{10020} \); B is the sums of the complex vectors \( h_{21000}, h_{10110}, h_{30000}, h_{10200}, h_{10020} \) produced by each sextupoles in the arcs when the phase advances of the TME cell are \( \Delta \mu_x = 0.5833 \times 2\pi \) and \( \Delta \mu_y = 0.25 \times 2\pi \).
It was complicated to find a zero solution for all constraints even when assigning significant weight factors for "stiff constraints", because the arc lattice is highly symmetric. The strength of SD sextupoles from different families is slightly different. The same situation is for the SF sextupoles. On average, the difference is about 1%. The resulting tune shifts with amplitude for the $\Delta \nu_x^{\text{TME}} = 0.5833$, $\Delta \nu_y^{\text{TME}} = 0.25$

$$\frac{\partial \nu_x}{\partial \epsilon_x} = 1.3446 \times 10^5, \quad \frac{\partial \nu_x}{\partial \epsilon_y} = \frac{\partial \nu_y}{\partial \epsilon_x} = -6.5682 \times 10^4, \quad \frac{\partial \nu_y}{\partial \epsilon_y} = 1.7974 \times 10^7$$

The Fourier harmonic $h(s)_{jk1m}e^{i(\alpha_{jk1m}(p)}$ produced by each sextupole in the arc can be represented geometrically as a vector on the complex plane. The integrals of all the complex vectors representing first order geometric aberrations $h_{21000}$, $h_{10110}$, $h_{30000}$, $h_{10200}$, $h_{10020}$ along the lattice become the vector sums, as shown in Fig 5.5B. Here we consider the damping ring composed of the two arcs only (without suppressors and straight sections) which includes of 8 identical achromats. The total chromaticity is equal to $\Delta \xi_x = 24.36$, $\Delta \xi_y = 20.72$ (the same value with reverse sign are induced by straight sections and suppressors).

The column B on Fig. 5.5 correspond to the individual vectors entering in vector sum along the 8 subsequent achromats. An overall horizontal and vertical phase advance of the two arcs are $\Delta \mu_x^{\text{arcs}} = 56 \times 2 \pi$ and $\Delta \mu_y^{\text{arcs}} = 24 \times 2 \pi$, respectively. The values of these sums at the nearest integer, 3rd-integer and 3rd-coupled resonances are inscribed on the plots. The spectrum $h(p)$ of five resonance driven terms is shown in Fig 5.5A. As one can see from the plots, the strength of 3rd-order coupled $\nu_x \pm 2 \nu_y$ and 3rd-order integer $3 \nu_x$, $\nu_x$ resonances is strongly enhanced at the harmonic numbers $p = 0$, $p = \pm 96$ and $p = \pm 192$.

5.5 Dynamic aperture

In our consideration we quote the dynamic aperture in terms of $\sigma_{inj}$ of the injected beam with normalized emittances of $\gamma \epsilon_x = 63 \mu$m and $\gamma \epsilon_y = 1.5 \mu$m. The dynamic aperture of the damping ring without dispersion suppressors and straight sections is shown in Fig 5.6. The TMEs cell are tuned to $\Delta \mu_x = 0.5833 \times 2 \pi$ and $\Delta \mu_y = 0.25 \times 2 \pi$.

In the CLIC damping ring, betatron tunes can be changed by the matching sections which connect the arc and wiggler straight section. It is possible to vary the machine betatron tunes by this section without optics disturbance in the arcs and wiggler sections. In fact, the highly symmetric achromatic lattices in the arc makes it possible to compensate the first order chromaticity and cancel the first order geometric aberration by two sextupole families. The small periodicity comes from the inclusion in our consideration of the two very long dispersion-free straight sections. In other words, we have two super periods with mirror symmetry which must be matched between each other from the nonlinear optics point of view to avoid dynamic aperture degradation. A few families of harmonic sextupoles or octupoles inserted in the straights may enlarge the dynamic aperture, since it is complicated to compensate all second order sextupole aberrations only in the arc.

One of the ways of cancellation between two super periods is to adjust the first order transformation matrix $R(s)$ between the end of the last achromat located in the first arc and the beginning of the next achromat located in the second arc to $+I$. In this case, suppressors and FODO wiggler sections become transparent for on-momentum particles from the linear motion point of view. This approach was used in our damping ring design.
Figure 5.6: The on-momentum dynamic aperture without straight sections when phase advance of TME cell $\Delta \mu_x = 0.5833 \times 2\pi$, $\Delta \mu_y = 0.25 \times 2\pi$, overall phase advance of the two arcs $\mu_x^{arcs} = 56 \times 2\pi$, $\mu_y^{arcs} = 24 \times 2\pi$.

Figure 5.7: The dynamic aperture of the damping ring. The working point $\nu_x = 69.82$, $\nu_y = 33.7$. 

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To stay away from the integer betatron tunes which results from the $+I$ matching, the phase advance of the TME cell was slightly detuned to the $\Delta \mu_x = 0.58146 \times 2\pi$ and $\Delta \mu_y = 0.2468 \times 2\pi$. The resulting horizontal dynamic aperture for the entire ring is shown in Fig 5.7. A dynamic aperture of $7\sigma_{inj}^x$ horizontally and $14\sigma_{inj}^y$ vertically in terms of injected beam size can be obtained for the CLIC damping ring lattice.

The limits in the on-momentum dynamic aperture can be explained by the tune shifts with amplitude. Even after optimization of the lattice, the tune shifts with amplitude are still large. Further work is required to refine the sextupole positions in the arcs to minimize the tune shifts with amplitude. In complex lower symmetry lattices there can be many families and now with the drive towards minimum emittance solutions these are often in a region with significant dispersion (this blurs the distinction between chromatic and harmonic families).
Chapter 6

Nonlinearities induced by the short period NdFeB permanent wiggler and their influence on the beam dynamics

6.1 Review of wiggler magnet technologies and scaling law

A qualitative list of the advantages and disadvantages of the various wiggler magnet technologies which can be applied for the CLIC damping ring is given in the Table 6.1 below. All five magnet technologies, namely electromagnet, permanent magnet, hybrid permanent magnet, hybrid electromagnet or superconducting can be considered for the CLIC damping ring. Searching for the optimum wiggler design we took into account the following requirements:

• The wiggler magnetic parameters have to provide the required damping rate and final equilibrium emittances.

• The wiggler design should be simple in its construction, adjustment and maintenance.

• The cost efficiency is taken into account in the selection of the wiggler design because of the great number of the wigglers in the damping ring.

Assuming the planar wiggler design for each type of technology, the peak magnetic field $\hat{B}_w$ on axis is related with the gap $g$ and wiggler period $\lambda_w$ according to the fit given by K. Halbach [86]:

$$\hat{B}_w = a \exp \left[ b \frac{g}{\lambda_w} + c \left( \frac{g}{\lambda_w} \right)^2 \right]$$  \hspace{1cm} (6.1)

where both $\hat{B}_w$ and $a$ are expressed in units of Tesla and $b$ and $c$ are dimensionless. These parameters depend on the wiggler performance and materials used in the magnet. The coefficients $a$, $b$, $c$ summarized in Table 6.2 have been computed by P. Elleaume [87] using a 3D magnetostatic code.
Table 6.1: Wiggler magnet technologies for producing a high field with short period.

<table>
<thead>
<tr>
<th>Technology</th>
<th>Advantages</th>
<th>Disadvantages</th>
</tr>
</thead>
<tbody>
<tr>
<td>Electromagnet</td>
<td>Field tuning flexibility; Radiation hardness; Field stability</td>
<td>Power consumption; Low field (&lt; 1.7 T) at short wiggler period (7-10 cm)</td>
</tr>
<tr>
<td>Pure Permanent Magnet</td>
<td>Does not require power; Short wiggler period (7-10 cm)</td>
<td>Radiation damage; Field varies with temperature; No field tuning flexibility; Weak max field (&lt; 1.7 T)</td>
</tr>
<tr>
<td>Hybrid Permanent Magnet</td>
<td>Does not require power; Short wiggler period; Magnetic field &gt; 1.7 T can be achieved</td>
<td>Radiation damage; Field varies with temperature; No field tuning flexibility</td>
</tr>
<tr>
<td>(combination of permanent magnet blocks and high saturation steel)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Hybrid Electromagnet</td>
<td>Temperature stability better than for PPM; Field tuning flexibility (typically about 25 %); Magnetic field &gt; 1.7 T can be achieved</td>
<td>Radiation damage;</td>
</tr>
<tr>
<td>(combination of electromagnets &amp; permanent magnets)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Superconducting</td>
<td>High field at short wiggler period; Field stability</td>
<td>Cryogenic infrastructure</td>
</tr>
</tbody>
</table>
Table 6.2: Fit coefficients $a$, $b$ and $c$ defining the peak field $\hat{B}_w$ as a function of the ratio $g/\lambda_w$ in Eq. 6.1 for the different kinds of planar wigglers.

<table>
<thead>
<tr>
<th>Model</th>
<th>Technology</th>
<th>$a$</th>
<th>$b$</th>
<th>$c$</th>
<th>Gap range</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>PPM NdFeB</td>
<td>2.076</td>
<td>-3.24</td>
<td>0</td>
<td>$0.1 &lt; g/\lambda_w &lt; 1$</td>
</tr>
<tr>
<td>B</td>
<td>Hybrid NdFeB &amp; vanadium permendur</td>
<td>3.694</td>
<td>-5.068</td>
<td>1.520</td>
<td>$0.1 &lt; g/\lambda_w &lt; 1$</td>
</tr>
<tr>
<td>*</td>
<td>Hybrid SmCo &amp; vanadium permendur</td>
<td>3.333</td>
<td>-5.47</td>
<td>1.8</td>
<td>$0.07 &lt; g/\lambda_w &lt; 0.7$</td>
</tr>
<tr>
<td>B</td>
<td>Hybrid NdFeB &amp; iron</td>
<td>3.381</td>
<td>-4.730</td>
<td>1.198</td>
<td>$0.1 &lt; g/\lambda_w &lt; 1$</td>
</tr>
<tr>
<td>C</td>
<td>Superconducting, gap=12mm</td>
<td>12.42</td>
<td>-4.790</td>
<td>0.385</td>
<td>$12\text{mm} &lt; \lambda_w &lt; 48\text{mm}$</td>
</tr>
<tr>
<td>C</td>
<td>Superconducting, gap=8mm</td>
<td>11.73</td>
<td>-5.52</td>
<td>0.856</td>
<td>$8\text{mm} &lt; \lambda_w &lt; 32\text{mm}$</td>
</tr>
<tr>
<td>D</td>
<td>Electromagnet, gap=12mm</td>
<td>1.807</td>
<td>-14.30</td>
<td>20.316</td>
<td>$40\text{mm} &lt; \lambda_w &lt; 200\text{mm}$</td>
</tr>
</tbody>
</table>

* The fit produced by K. Halbach for the hybrid samarium cobalt & vanadium permendur wiggler design [85], [88] where the remanent field is 0.9 T.

Figure 6.1: Peak field versus gap/period approximated by Eq. 6.1 with parameters taken from Table 6.2.

The peak fields as a function of the ratio $g/\lambda_w$, according to Table 6.2 and Eq. (6.1), are presented in Fig. 6.1. The simulations were done for the commonly used wiggler designs shown in Fig. 6.2. Using the parameters $a$, $b$, $c$ facilitates estimating the limit of the peak field and choosing the proper wiggler technology for a particular application without the
need of a 3D field computation.

Figure 6.2: Commonly used magnetic design and dimensions of the wigglers based on A: pure permanent magnet technology, B: hybrid permanent magnet technology, C: superconducting technology, D: electromagnet technology. Red arrows indicate current, blue arrows are magnetization.

K. Halbach produced a similar fit [86]. The coefficients $a$, $b$, $c$ computed by him are slightly different from the coefficients summarized in Table 6.2. For example, for hybrid NdFeB and vanadium permendur [89] wiggler the coefficients are $a = 3.44$, $b = -5.08$ and $c = 1.54$, which gives a smaller value of the peak field $\tilde{B}_w$ at $0.07 < g/\lambda_w < 0.7$ as shown in Fig. 6.1. Probably the difference is explained by different sizes of the magnets and poles used for the field computation. The details of the wiggler designs shown in Fig. 6.2 are briefly described below.
Model A: Pure permanent magnet (PPM) wiggler

The PPM wigglers are assembled (without steel poles) by permanent magnet blocks made of NdFeB, SmCo$_5$ or Sm$_2$Co$_{17}$ material. The total height of the block is usually equal to half a wiggler period and the horizontal width is equal to one period. This choice is optimum with respect to cost. For example, only 4% extra peak field can be obtained if the height of the magnet blocks is doubled.

Maximum achievable amplitude of the fundamental sinusoidal component of the field for the infinitely long PPM planar wigglers can be estimated from the scaling law [89, 90]

$$B_w \approx 2B_r e^{-\pi g/\lambda_w} \sin(\pi/N_b) \left[1 - e^{-2\pi h/\lambda_w}\right]$$  \hspace{1cm} (6.2)

where $B_r$ is the remanent field, $N_b$ is the number of rectangular uniformly magnetized blocks per either top or bottom parts of the wigglers period ($N_b = 4$ for the case illustrated in Fig. 6.2A), $h$ is the height of the permanent magnet block. The equations are valid if the pole width is greater than the gap. The remanent field up to 1.4 T can be achieved by using NdFeB alloy while SmCo$_5$ and Sm$_2$Co$_{17}$ alloys can be magnetized only to 0.9 - 1.01 T and 1.04 - 1.12 T respectively.

Wigglers constructed by PPM or hybrid PM technology are sensitive to radiation in electro-magnetic showers. Some tests have shown that the alloys based on samarium and cobalt (SmCo$_5$ and Sm$_2$Co$_{17}$) have a higher resistance to radiation damage [91, 92].

In permanent magnets, the field stability is generally limited by the temperature coefficient of the remanence. For the NdFeB materials the change of remanent field with temperature (temperature coefficient) is $\Delta B_r/B_r = -0.12$ % per 1°C while SmCo$_5$ and Sm$_2$Co$_{17}$ materials have better temperature properties, namely $\Delta B_r/B_r = -0.06$ % and $\Delta B_r/B_r = -0.04$ % per 1°C respectively [93]. The temperature stability can be improved by introducing small correction electromagnets in the permanent magnets blocks. By controlling the current of these electromagnets with the aid of one or several temperature sensors mounted on the permanent magnet blocks, improvement of the temperature stability by up to a factor ten can be achieved. The main drawback in using Sm$_2$Co$_{17}$ instead of NdFeB is a lower remanent field and correspondingly poorer magnetic properties. Another possibility of thermal correction is based on combining two types of permanent magnet materials with different temperature coefficient of $B_r$ [94].

Model B: Hybrid permanent magnet (HPM) wiggler

A larger magnetic field can be reached by combination of PMs and iron poles. It is clear that the peak field of the HPM wiggler is higher than that of the PPM wiggle (see Fig. 6.1) because an iron pole concentrates the flux lines produced by the PM. However, the HPM wigglers usually use nearly three times more volume of permanent magnet than the PPM wigglers. As it can be seen from Fig. 6.1, a slightly higher peak field is obtained for poles made of vanadium permendur (a high saturation cobalt steel) instead of simple iron. The dimensions of permanent magnets and poles presented in Fig 6.2B for the HPM wiggler were optimized to maximize the peak field. At small values of $g/\lambda_w$, the field produced by the hybrid wiggler can be enhanced, if additional small magnet blocks are placed on each lateral side of the pole. Using such extra magnets, a peak field of 3.13 T has been reached for the ESRF asymmetric HPM wiggler [95] at the ratio $g/\lambda_w = 0.05$ ($g = 11$ mm).
**Model C: Superconducting short period wiggler**

The horizontal width of the superconductor has to be equal to at least twice the wiggler period to reach the maximum field. The cross-section of the superconducting coil is normalized to the wiggler period with ratio of $0.5\lambda_w$ and $0.35\lambda_w$ in the vertical and longitudinal direction, respectively, as shown in Fig 6.2C.

At the present moment, niobium-titanium NbTi (9.2K, 14.5T) and Nb$_3$Sn (18.3K, 22.5T) are two commercially available superconductors which are offered by manufacturers worldwide. The other superconducting materials such as brittle intermetallic compound Nb$_3$Al (18.8K, 29.5T), Nb$_3$Ge (23.2K, 37T), V$_3$Ga (15K, 22T) and Chevrel phase compounds like PbMo$_6$S$_8$ (14K, 60T) which show advantages compared with NbTi as regards to the critical field $B_{c2}(0)$ at $T_c=0$ K and the critical temperature $T_c(0)$ at $B_{c2}=0$ T (parameters in the brackets) are produced in very small quantities, since it is difficult to develop an economical production method for these alternative superconductors.

Based on the standard LHC-type Cu:NbTi superconductor cable [96, 97] used for the LHC main quadrupoles, the maximum field in the SC coil of the wiggler shown in Fig 6.2C can be estimated as a function of wiggler period and gap. The main characteristics of the strand of the LHC superconductor cable are the following [97]:

- Diameter after coating, $D_s$: $0.825 \pm 0.0025$ [mm]
- Copper to superconductor ratio, $R_{Cu/SC}$: $1.95 \pm 0.05$
- Filament diameter, $D_f$: $6$ [$\mu$m]
- Number of filaments, $N_f$: $\sim6500$
- Critical current density of filaments at 6 T and 4.2 K: $2000$ [A/mm$^2$]
- Critical current density of filaments at 5 T and 4.2 K: $2550$ [A/mm$^2$]

Usually NbTi filaments are embedded into a copper matrix. In superconducting regime, a current is flowing through the NbTi filaments only. If the value of the current density in the NbTi filaments exceeds the critical value

$$J_c = \frac{J_0}{1 + |B|/B_0} \bigg|_{T=\text{const}} \quad \Rightarrow \quad B = B_0 \left( \frac{J_0}{J_c} - 1 \right) \bigg|_{T=\text{const}} \quad (6.3)$$

at a given field $B$ and temperature $T$, the superconductor becomes normal and the current is shared between the copper matrix and now resistive NbTi filaments. The Eq. (6.3) is an empirical relation stated by Kim [98] for the low-field application where $J_0$ and $B_0$ are constants which are determined by the production process rather than by the intrinsic properties of NbTi. Taking the critical current density for standard LHC cable at the field of 5 T and 6 T (see parameters listed above), the constants $J_0$ and $B_0$ at the temperature of 4.2 K are defined by nonlinear fitting as $J_0 = 26.577$ kA/mm$^2$ and $B_0 = 0.512$ T respectively. Cu and NbTi are non-magnetic materials which have $\mu_r \sim 1$.

Using a 3D magnetic code, for the case of constant gap of 12 mm, the critical current density $J_c$ averaged over the whole cross-section $(0.5 \times 0.35 \times \lambda_w^2)$ of the coil was computed as a function of ratio between gap and wiggler period. The result is shown in Fig. 6.3 (left plot). The $J_c$ dependence on $g/\lambda_w$ at $g = 12$ mm can be approximated as $J_c(\text{kA/mm}^2) = 2.395 - 5.924/(2.883 + g/\lambda_w)$. The packing factor $P_f$ (the total cross-section of cables divided by cross-section of coil) was chosen to 0.72. The ratio between the current density in the filaments to the current density averaged over the whole cross-section of the coil is defined by the coefficient $D_s^2/(N_fD_f^2P_f) = 4.04$ (see the cable parameters listed above).
Defining the coordinate origin of longitudinal axis \( s \) as shown in Fig. 6.2C, the magnetic field takes peak value at the planes \( s = 0.5n\lambda_w \) and the field is zero at the planes \( s = (0.25 + 0.5n)\lambda_w \), where \( n \) denotes an integer number. The operating current density \( J_{op} \) was chosen as 85 \% of the critical current \( J_c \), which is quite typical. For the cases \( J_{op}/J_c = 0.85 \) and \( J_{op}/J_c = 1 \), the peak field as a function of \( g/\lambda_w \) is shown in Fig. 6.3 (right plot) as the blue and black solid lines, respectively. Using the fit given by Eq. (6.1), the peak field dependence can be approximated as

\[
\hat{B}_w = 12.249 \exp \left[-5.356 \frac{g}{\lambda_w} + 0.587 \left( \frac{g}{\lambda_w} \right)^2 \right], \quad \text{for } J_{op}/J_c = 0.85
\]

\[
\hat{B}_w = 10.412 \exp \left[-5.356 \frac{g}{\lambda_w} + 0.587 \left( \frac{g}{\lambda_w} \right)^2 \right], \quad \text{for } J_{op}/J_c = 1
\]

(6.4)

The coefficients \( a, b, c \) of the fit given by Eq. (6.4) are very close to Elleaume’s coefficients presented in Table 6.2 for the same geometrical model of SC wiggler. The maximum field inside the SC coils is shown by the red solid line in Fig. 6.3 (right plot).

Finally, we consider SC coils in a helium \((^4\text{He})\) bath with temperature \( T_b = 4.2 \) K. If the temperature of \(^4\text{He}\) exceeds the limit \( T_{cs} \) given by [99]

\[
T_{cs} = T_b + \left( T_c(0) \left[ 1 - \left( \frac{B}{B_{c2}(0)} \right) \right]^{0.59} - T_b \right) \left( 1 - \frac{J_{op}}{J_c} \right)
\]

(6.5)

then the superconducting regime is broken. For NbTi, \( T_c(0) \) (at \( B = 0, I = 0 \)) is 9.2 K and \( B_{c2}(0) \) is 14.5 T (at \( T = 0, I = 0 \)). The \( T_{cs} \) is equal to 4.65 K at the highest field of 6.5 T inside CS coil, which leaves a temperature margin of \( \Delta T = 0.45 \) K, before the magnet quenches.

**Model D: Electromagnet wiggler**

Room-temperature electromagnet technology is much less efficient than HPM, PPM or superconducting technologies for producing a high field at short wiggler period and gap. The peak field shown in Fig. 6.1 for the electromagnet wiggler was simulated by the RADIA code for the wiggler model presented in the paper [87]. In this model, the horizontal width of the yoke, the height of the coil and the average current density in the coil were optimized to 50 mm, 100 mm and 2 A/mm\(^2\), respectively.
6.2 Tentative design of hybrid permanent NdFeB wiggler for the CLIC damping ring

A tentative design of the NdFeB hybrid permanent wiggler for the CLIC damping ring is based on the wiggler design for the PETRA-3 ring [100]. The parameters of the PETRA-3 wiggler [101] (wiggler period, gap, field amplitude) were re-optimized to meet CLIC damping ring requirements. An optimized design of the NdFeB permanent wiggler for the CLIC damping ring is shown in Fig. 6.4 and the corresponding wiggler parameters are summarized in Table 6.3.

Table 6.3: Wiggler parameters of the NdFeB wiggler

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Field amplitude</td>
<td>1.7 T</td>
</tr>
<tr>
<td>Period of the wiggler</td>
<td>10 cm</td>
</tr>
<tr>
<td>Number of periods</td>
<td>20</td>
</tr>
<tr>
<td>Magnetic gap of the wiggler</td>
<td>12 mm</td>
</tr>
<tr>
<td>Pole width</td>
<td>60 mm</td>
</tr>
<tr>
<td>Magnet material</td>
<td>Nd-Fe-B</td>
</tr>
<tr>
<td>Pole material</td>
<td>Permendur</td>
</tr>
</tbody>
</table>

Note that the wedge-shaped pole design of the wiggler was chosen. The main advantage of this design is the absence of the electromagnetic coupling between adjacent poles. The choice of the wedge-shaped poles instead of rectangular-shape poles results in a substantial decrease in the stray magnetic flux. This feature simplifies the adjustment procedure. Changing the vertical gap by a special ”bolt-corrector” (see Fig. 6.4), the wiggler field amplitude is varied. As one can see from Fig. 6.4, one wiggler period is made from the four ”front” PM blocks, \( N_b = 4 \).

Maximum achievable peak field amplitude of the NdFeB wiggler versus pole gap at different period lengths [102] was computed by 3D code MERMAID [103] as shown in Fig. 6.5 (left plot). Taking into account the remanent field of 1.35 T, the peak field at the gap \( \leq 12 \text{ mm} \) is in a good agreement with Halbach scaling Eq. (6.1) with \( a = 3.44, b = -5.08 \) and \( c = 1.54 \) (see also Fig. 6.1). Nevertheless, at the gap of 16 mm, the peak field from the numerical simulations is about 0.1 T less than, the one predicted by Eq. (6.1). The peak field is linear for reasonably small changes of the pole gap. Distribution of magnetic field (field map) for the HPM NdFeB wiggler with \( \lambda_w = 10 \text{ cm} \) and \( g = 12 \text{ mm} \) was computed by the 3D code MERMAID. This code performs fast calculations of the horizontal, vertical, and longitudinal field components at each point on a rectangular grid with a precision of \( 10^{-3} \).

It is impossible to maintain the wiggler field amplitude for decreasing period, keeping just the same vertical gap since the magnetic induction in the pole tips reaches its maximum value, which for permendur is 21-22 kG. The dependence of the maximum on-axis field versus the wiggler period at fixed gap of 12 mm is shown in Fig. 6.5 (right plot). A decrease in the period with the same field amplitude turns out to be possible only in the case of a substantial over-expenditure of permanent magnets. So, for example, by a two-fold increase in the volume of the magnetic material for the optimized design of the CLIC HPM wiggler we can decrease the wiggler period only by 10%.
Figure 6.4: One period of the NdFeB wiggler [102].
Figure 6.5: Peak field amplitude vs. pole gap for different wiggler periods (left plot); peak field amplitude vs. period length at a gap of 12 mm (right plot).

### 6.3 SR power and absorption

An effective collimation system in the wiggler straight sections is very important. Because of the large synchrotron radiation power an interception strategy has to be studied. A HPM wiggler would require upstream collimation to limit the radiation within the straight wiggler sections. The parameters of synchrotron radiation from the HPM NdFeB wiggler presented in Sec. 6.2 are summarized in the Table 6.4 below. The SR power generated by one wiggler module is directly proportional to the average beam current in the damping ring. The maximum beam current corresponding to the maximum number of bunch trains which can be stored in the damping ring is $I = N_{b}^{max} k_{bt} N_{bp} 1.6 \times 10^{-19}/T_{0}[s] = 0.52$ A for the design parameters listed in Table 4.8 and 4.9 for the RING 1 (bunch population $N_{bp} = 2.56 \times 10^{9}$, No. of bunches per train $k_{bt} = 110$, maximum number of bunch trains $N_{b}^{max} = 14$ and revolution time per one turn $T_{0} = 1.216 \mu s$). As seen from Table 6.4, the SR power $P_{T}$ generated by one wiggler module with length of 2 m is equal to 11.18 kW. Taking into account that the damping ring includes 76 wigglers in the two straight sections, the total radiation power from all wigglers is equal to 849.6 kW.

The angular distribution of SR power $\frac{dP}{d\Omega}$ irradiated by the wiggler is estimated by the following basic formulas [104]:

$$\frac{dP}{d\Omega} = \frac{d^2P}{d\theta d\psi} = P_{T} \frac{21 \gamma^2}{16 \pi K} G(K) f_{K}(\gamma \theta, \gamma \psi)$$ (6.6)
where

\[
G(K) = K \frac{K^6 + \frac{24}{7} K^4 + 4 K^2 + \frac{16}{7}}{(1 + K^2)^{7/2}} \quad (6.7)
\]

When an electrons follow a sinusoidal trajectory and the \( K \) parameter is large \((K > 10)\), the function \( f_K \) can be estimated with good accuracy by

\[
f_K(\gamma \theta, \gamma \psi) = \sqrt{1 - \left(\frac{\gamma \theta}{K}\right)^2} \left[ \frac{1}{(1 + (\gamma \psi)^2)^{5/2}} + \frac{5(\gamma \psi)^2}{7(1 + (\gamma \psi)^2)^{7/2}} \right] \quad (6.8)
\]

where \( \theta \) and \( \psi \) are angles in horizontal and vertical plane, respectively.

Three possible methods for absorption of the SR power can be applied [105].

- Distributed absorbers.
- Few long absorbers.
- Poly-line trajectories.

The schematic view for each approach is illustrated in Fig. 6.6.

![Diagram of three possible approaches for the absorption of SR power in the damping ring.](image)

Figure 6.6: Three possible approaches for the absorption of SR power in the damping ring.

The main disadvantage of the third method (poly-line trajectories) is that a few small achromatic bends \((\sim 1^\circ)\) of beam trajectory, provided for example by DBA cell in the dispersion free straight section, are needed to let out the radiation to an absorbers.
Using several long absorbers which can be placed instead of some wigglers, results in overheating of the vacuum chamber between neighbouring long absorbers and also yields a big power of SR at the terminal absorber. If the 12th, 24th and 36th wiggler are replaced by a long absorber, than integrated value of SR power deposited on the vacuum chamber is about 600 W/m which is not acceptable.

From our point of view, the regularly distributed small absorbers is the more preferable variant, though it still leads to quite significant power in the terminal absorber. This variant is considered below.

SR power loads on wiggler vacuum chambers and regularly distributed copper absorbers were simulated for the closed orbit distortion of 100 μm. The simulation is based on Eqs. (6.6–6.8). In the straight FODO section each absorber is located between a wiggler and a quadrupole as shown in Fig. 6.6. An absorber in front of defocusing quadrupoles has the vertical aperture equal to 6 mm, but the vertical aperture of absorbers located in front of focusing quadrupoles is 4 mm, as it is sketched in Fig. 6.7. The horizontal aperture for all absorbers is identical, and equal to 60 mm. In total, 38 absorbers are located in one straight section. In Fig. 6.7 the wigglers are indicated by yellow rectangles and the absorbers by blue vertical lines. The red thick lines correspond to the central rays from the wigglers and the red dashed thin lines to the divergent rays spread from the central ray by angle of ±1/γ.

Such configuration of the regular distributed absorbers provides absorption of 334.5 kW of SR power per the straight section. The rest of the SR power, 90.3 kW, will be taken up by the terminal absorber placed at the end of the straight section. On average, the absorbers with vertical aperture of 4 mm and 6 mm absorb 13.4 kW and 4.2 kW of SR power respectively as shown in Fig. 6.8a (lower plot). The inward cavity of the absorbers have a wedge-shape. The power density distribution on the surface of the absorbers No.25 (vertical aperture of 4 mm) and No.20 (vertical aperture of 6 mm) is shown in Fig. 6.8b. The density reaches the maximum value of 200 W/mm² at the edge of the aperture. The integrated SR power load for absorbers No.25 and No.20 is 16 kW and 5 kW, respectively.

A small fraction of power hits the vacuum chamber. Near the absorbers with aperture of 4 mm the power deposited on the vacuum chamber is maximum, but the value of power density in this place does not exceed 0.63 mW/mm² that corresponds to 12 W/m. The integrated value over the vacuum chamber of the straight section is equal to 6 W/m as it could be seen in Fig. 6.8a (upper plot).
Figure 6.8: a) Total power loads for vacuum chambers (the upper plot) and absorbers (the lower plot); b) distribution of power density in the absorber of No.25 with vertical aperture of 4 mm (the upper plot) and absorber of No.20 with vertical aperture of 6 mm (the lower plot).

Considering the regular absorbers with vertical aperture of 6 mm and 8 mm instead of 4 mm and 6 mm, respectively, SR power of 320 kW is absorbed by 38 absorbers located in the straight section. However, the maximum value of power deposited on the vacuum chamber can exceed 240 W/m (13 mW/mm²) which is not acceptable.

6.4 Fitting the wiggler field

A simulation of nonlinearities in the wiggler field is often done by inserting thin multipoles throughout the wiggler. For example, one wiggler period might be modelled as two combined-function bends with positive and negative polarity separated from each other by a quarter of a wiggler period $\lambda_w$. Thin octupole lenses are placed at the ends of each bending magnets. This modelling can be performed by using standard MAD elements.

The main disadvantage of this technique is that the resulting field is not consistent with Maxwells equations and that the position, order and strength of the multipoles can not be consistent with the actual situation.

A magnetic field map for the wiggler can be computed using a modelling code such as TOSCA, RADIA, OPERA or Mermaid. The Mermaid code was used for the calculation of the magnetic field map in the HPM NdFeB wiggler that, was presented in Sec. 6.2. For a given wiggler design Mermaid calculates the horizontal, vertical, and longitudinal field components at each point on a rectangular grid. For analyzing the particle dynamics in the damping wiggler, we can compute the amplitude of various field modes to estimate their contribution to the limit on the dynamic aperture. We will consider only the wiggler nonlinearities which are included in the field map of the design wiggler model. For this purpose, an analytic series of the field is fitted to the numerical field map which reduces the field map to a set of coefficients. The analytic series is then used to generate a dynamical map for particle tracking through the wiggler.
6.4.1 Magnetic field model in Cartesian expansion

Assuming reflection symmetry in each of the three major co-ordinate planes the multipole wiggler field can be described by setting the longitudinal components of the vector potential equal to zero, which is allowed by gauge invariance. In this case the magnetic vector potential in the wiggler field can be written as \[106, 107\]:

\[
A_x = \sum_{m,n} \frac{1}{nk_z} \cos(mk_x x) \sin(nk_z z) \cosh(k_{y,nn} y)
\]

\[
A_y = \sum_{m,n} \frac{mk_x}{nk_z k_{y,nn}} \sin(k_x x) \sin(nk_z z) \sinh(k_{y,nn} y)
\]

\[
A_z = 0 \quad (6.9)
\]

The three-dimensional magnetic field for a planar horizontal wiggler derived from the vector potential as \( \mathbf{B} = \nabla \times \mathbf{A} \) is expressed in the following form:

\[
B_x = -\sum_{m,n} \frac{mk_x}{k_{y,nn}} \sin(mk_x x) \cos(nk_z z) \sinh(k_{y,nn} y)
\]

\[
B_y = \sum_{m,n} c_{mn} \cos(mk_x x) \cos(nk_z z) \cosh(k_{y,nn} y)
\]

\[
B_z = -\sum_{m,n} \frac{nk_z}{k_{y,nn}} \cos(mk_x x) \sin(nk_z z) \sinh(k_{y,nn} y) \quad (6.10)
\]

where \( k_z = 2\pi/\lambda_w \), and \( \lambda_w \) is the wiggler period. Maxwells equations are satisfied if we impose the conditions:

\[
k_{y,nn}^2 = m^2 k_z^2 + n^2 k_z^2
\]

The assumed symmetry conditions have made the field periodic in \( x \). For an ideal wiggler with infinitely wide pole, \( k_x \) tends to zero, and the field is independent of \( x \). However if the field is known between limits \( \pm L_x \), we can choose the horizontal periodicity as \( k_x = 2\pi/L_x \). If this limit is large compared to the region of interest for beam dynamics, the field periodicity can be extended as \( k_{x,m} = mk_x \).

By using a 2-dimensional Fourier transform, the coefficients \( c_{mn} \) can be derived from field data in the \( x - z \) plane. However the coefficients \( c_{mn} \) determined from the Fourier transform will not correspond exactly to the real field because we use a limited range of field data and a finite number of modes for the Fourier transform calculation. This will result in some divergence between the fitted field and the real field map, especially far from the longitudinal axis of symmetry. Small corrections to the higher order coefficients (with large \( m \) and \( n \)) can improve the correspondence between the fitted field and the field map in the vertical direction without degrading the fit in the horizontal and longitudinal planes. In detail, such technique is described in the paper \[107\]. The main disadvantages of the Cartesian expansion are:

- the Cartesian expansion (6.10) assumes that the field is periodic in the horizontal co-ordinate, which is generally not the case.
- a large number of modes are needed to obtain good accuracy.
- procedure can be time consuming

6.4.2 Magnetic field model in cylindrical expansion

The description of the magnetic field in a current-free region is most conveniently carried out in terms of a scalar potential \( \Psi \) obeying the Laplace equation \( \nabla^2 \Psi = 0 \). Solving Laplace equation in
The scalar potential \( \Psi \) can be defined as:

\[
\Psi = \sum_{mn} I_m(nk_z \rho) \cos(nk_z z) \left[ b_{mn} \sin(m\phi) + a_{mn} \cos(m\phi) \right]
\]  

(6.11)

where \( I_m(x) \) is a modified Bessel function. The coefficients of \( \sin \) and \( \cos \) correspond to the normal and skew components, respectively. The skew components are negligible, if the wiggler does not have any alignment errors. In our further consideration we assume that only normal field components are present. The corresponding expressions for the magnetic field in cylindrical coordinates, which satisfies the equation \( \mathbf{B} = \nabla \Psi \), are given by

\[
B_\rho = \sum_{mn} nk_z b_{mn} I'_m(nk_z \rho) \sin(m\phi) \cos(nk_z z)
\]

\[
B_\phi = \sum_{mn} m b_{mn} I_m(nk_z \rho) \cos(m\phi) \cos(nk_z z)
\]

\[
B_z = -\sum_{mn} nk_z b_{mn} I_m(nk_z \rho) \sin(m\phi) \cos(nk_z z)
\]  

(6.12)

Having numerical field data over the surface of a cylinder coaxial with the longitudinal wiggler axis the coefficients, \( b_{mn} \) can be found simply from a two-dimensional Fourier analysis. For the minimization of fitting errors, the radius of the cylinder has to be as large as possible. Taking into account the exponential behavior of the modified Bessel function, fitting errors decrease towards the axis of the cylinder and increase away from the axis. Therefore, if the numerical field data away from the axis of the wiggler have been calculated or measured with a good accuracy, then it is preferable to take the surface with a large radius for the fit.

The cylindrical expansion (6.12) reflects the natural periodicity in the azimuthal coordinate. The cylindrical expansion can be converted into a Cartesian expansion with a good fit within the cylinder surface. For a given value of \( k_x \), the coefficients \( b_{mn} \) and \( c_{mn} \) are related by [108]

\[
b_{mn} = \frac{2(-1)^m m!}{(nk_z)^m} \sum_m \left( \sum_{q=0}^{m-1} \frac{k^m 2q-1 (m' k_x)^{2q}}{(2q)! (m - 2q)!} \right) c_{m'n}
\]

This expression allows one to calculate a set \( b_{mn} \) from a given set of \( c_{mn} \), or by a matrix inversion a set of \( c_{mn} \) from a given set of \( b_{mn} \). The main advantage of the cylindrical expansion is:

- Fourier analysis is more naturally done using cylindrical coordinate basis functions (natural periodicity in azimuthal coordinate is preserved).

6.4.3 Multipole expansion for the scalar potential and generalized gradients

The 3D multipole expansion can be easily converted into a power series in the radial variable \( \rho \) by using the Taylor expansion for \( I_m(x) \):

\[
I_m(x) = \sum_n \frac{1}{n!(n + m)!} \left( \frac{x}{2} \right)^{2n+m}
\]  

(6.13)

The substitution of the Taylor series (6.13) in the expression (6.11) and inversion of the order of the double summation yield the following expansion [109] for the magnetic field in cylindrical coordinates

\[
B_\rho = \sum_{m=1}^{\infty} \sum_{k=0}^{\infty} (-1)^k \frac{m!(2k + m)}{2^k k!(k + m)!} C_{m'}^{[2k]}(z) \rho^{2k+m-1} \sin(m\phi)
\]

107
\[
B_\phi = \sum_{m=0}^{\infty} \sum_{k=0}^{\infty} (-1)^k \frac{m!(2k+m)}{2^{2k}k!(k+m)!} C_m^{[2k]}(z) \rho^{2k+m-1} \cos(m\phi) \\
B_z = \sum_{m=1}^{\infty} \sum_{k=0}^{\infty} (-1)^k \frac{m!(2k+m)}{2^{2k}k!(k+m)!} C_m^{[2k+1]}(z) \rho^{2k+m} \sin(m\phi)
\]

where the functions \(C_m^{[2k]}\) are defined by
\[
C_m^{[2k]}(z) = \frac{(-1)^k}{\sqrt{2\pi}} \frac{1}{2^m m!} \sum_{p=-\infty}^{\infty} e^{2\pi i pz/\lambda_w} \left( \frac{2\pi p}{\lambda_w} \right)^{2k+m} b_{m,p}
\]
and \(b_{m,p}\) is calculated below. Note that \(C_m^{[2k+2]} = \frac{d^2}{d\phi^2} C_m^{[2k]}\) and \(C_m(z) = C_m^{[0]}(z)\). In the following, the functions \(C_m(z)\) will be referred to as "generalized gradients" [109].

It is easy to calculate the generalized gradients from numerical field data. Knowledge of one component of the magnetic field on the surface of a cylinder is sufficient to determine the entire field in the current-free region both inside and outside that surface. We suppose that the radial component \(B_\rho(\rho = R, \phi, z)\) is known on the cylindrical surface with radius \(R\). The Fourier series in terms of the azimuthal angle is given by
\[
B_\rho(\rho = R, \phi, z) = \sum_{m=0}^{\infty} B_m(R, z) \sin(m\phi)
\]
with the
\[
B_m(R, z) = \frac{1}{\pi} \int_0^{2\pi} \sin(m\phi) B_\rho(R, \phi, z) d\phi
\]

From Eq. (6.11), the relation \(\mathbf{B} = \nabla \Psi\) and Eq. (6.16), the coefficients \(b_{m,p}\) are found as
\[
b_{m,p} = \frac{1}{2\pi p I_m'(2\pi p R/\lambda_w)} \lambda_w e^{-i2\pi pz/\lambda_w} \int_0^{\lambda_w} e^{-i2\pi pz/\lambda_w} B_m(R, z) dz
\]

Insertion of Eq. (6.18) into the definition of the generalized gradients Eq. (6.15) gives the following result:
\[
C_m^{[k]} = \frac{1}{\lambda_w 2^m m!} \sum_{p=-\infty}^{\infty} i^k \left( \frac{2\pi p}{\lambda_w} \right)^{m+k-1} e^{-i2\pi pz/\lambda_w} \int_0^{\lambda_w} e^{-i2\pi pz/\lambda_w} B_m(R, z) dz
\]

The method described above yields a smooth representation of the numerical field data as will be seen in the next section. The Bessel functions \(I_m(x)\) grow exponentially for large arguments. The denominator \(I_m'(2\pi p R/\lambda_w)\) provides an effective high-frequency filter in the evaluation of the generalized gradients, i.e., the denominator \(I_m'(2\pi p R/\lambda_w)\) in the Eq. (6.19) acts as a filter that damps high frequency components of the magnetic field data. It reduces the numerical noise possibly present in the magnetic field data. The efficiency of filtering is enhanced with increasing value for the cylinder radius \(R\). The summary of this method is the following:

- high frequency components of magnetic field data are suppressed.

### 6.5 Analysis of field map for the NdFeB HPM wiggler design

In this section, the cylindrical expansion of a magnetic field (6.14) is fitted to the numerical field map computed for one wiggler period of the NdFeB HPM wiggler design. This design was described in the Section 6.2. The wiggler has 20 periods of 10 cm, peak field of 1.7 T and gap 12 mm.
A numerical field map in Cartesian co-ordinates was computed on cylinder surface coaxial to the longitudinal wiggler axis $z$ with the largest radius $R$ of 5 mm. Horizontal $B^\text{dat}_x$, vertical $B^\text{dat}_y$ and longitudinal $B^\text{dat}_z$ field components, which are evaluated on a cylindrical grid with azimuth and longitudinal step of $\Delta \phi = 5^\circ$ and $\Delta z = 1$ mm respectively, are provided as input data $B^\text{dat}_x(R, \phi_i, z_i) = B^\text{dat}_y \sin \phi_i + B^\text{dat}_z \cos \phi_i$, $B^\text{dat} \phi(R, \phi_i, z_i) = B^\text{dat}_y \cos \phi_i - B^\text{dat}_z \sin \phi_i$, and $B^\text{dat}_z(R, \phi_i, z_i) = B^\text{dat}_z$ to the code developed as part of this thesis to determine the generalized gradients (6.19). Taking Fourier integral (6.17) on $\phi$ from $B^\text{dat}_x(R = 5\text{mm}, \phi_i, z_i)$ up to mode number $m = 9$, we obtain the harmonics $B^\text{m}_i(R = 5\text{mm}, z_i)$. Inserting $B^\text{m}_i(R = 5\text{mm}, z_i)$ into Eq. (6.19), the generalized gradients and their derivatives are found. The resulting profiles of the generalized gradient $C^m_1(z)$ (that is equal to the on-axis vertical component of the magnetic field), $C^m_3(z)$, $C^m_5(z)$ and their derivatives are shown in Fig. 6.9.

In the ideal error-free design considered here the presence of even harmonics of $B^\text{m}_i(R = 5\text{mm}, z)$, as well all that of the skew components, is prevented by the anti-symmetry of the fields under rotation of 180° around the axis. Using the generalized gradients $C^k_m(z)$, now we can find via Eq. (6.14) the magnetic field components at any point $(\rho, \phi, z)$. For instance, the multipole expansion of $B_\rho$ through 5th order in $R$ is expressed by

$$B^\text{fit}_\rho = \left( C_1 - \frac{3R^2}{8} C_1^{[2]} + \frac{5R^4}{192} C_1^{[4]} \right) \sin \phi + \left( 3C_3 R^2 - \frac{5R^4}{16} C_3^{[2]} \right) \sin 3\phi + \left( 5C_5 R^4 \right) \sin 5\phi$$

(6.20)

The profiles for the odd-harmonics $B_m(R = 5\text{mm}, z)$ through $m = 5$ are plotted in Fig. 6.10. The azimuthal harmonics $B^\text{m}_m(R = 5\text{mm}, z)$ derived from the numerical field data by the Fourier integral (6.17) are shown in blue color in Fig. 6.10. The red color corresponds to the azimuthal harmonics $B^\text{fit}_m(R = 5\text{mm}, z)$ computed by generalized gradients $C^k_m$ in Eq. (6.20).

It is easy to convert the cylindrical field representation to the Cartesian form. From Eq. (6.14) the expressions for the horizontal, vertical and longitudinal magnetic field components through 4th order in Cartesian coordinates $B_x = B_\rho \cos \phi - B_\phi \sin \phi$, $B_y = B_\rho \sin \phi + B_\phi \cos \phi$ can be written as:

$$B_x = - \left( \frac{1}{4} C_1^{[2]} - 6C_3 \right) xy + \left( \frac{1}{48} C_1^{[4]} - \frac{3}{4} C_3^{[2]} + 20C_5 \right) x^3 y + \left( \frac{1}{48} C_1^{[4]} - \frac{1}{4} C_3^{[2]} - 20C_5 \right) x^3 y^3$$

$$B_y = C_1 - \left( \frac{1}{8} C_1^{[2]} - 3C_3 \right) x^2 - \left( \frac{3}{8} C_1^{[2]} + 3C_3 \right) y^2 + \left( \frac{1}{192} C_1^{[4]} - \frac{3}{16} C_3^{[2]} + 5C_5 \right) x^4 +$$

$$+ \left( \frac{1}{32} C_1^{[4]} - \frac{3}{8} C_3^{[2]} - 30C_5 \right) x^2 y^2 + \left( \frac{5}{192} C_1^{[4]} - \frac{5}{16} C_3^{[2]} + 5C_5 \right) y^4$$

$$B_z = yC_1^{[1]} - \left( \frac{1}{8} C_1^{[3]} - 3C_3^{[1]} \right) x^2 y - \left( \frac{1}{8} C_1^{[3]} - C_3^{[1]} \right) y^3$$

(6.21)
Figure 6.9: The generalized gradients $C_1(z)$, $C_3(z)$ 3rd (sextupole), $C_5(z)$ 5th (decapole) order and their derivatives. $C_1(z)$ is equal to the wiggler on-axis magnetic field.
Figure 6.10: The profiles of the azimuthal odd-harmonics $B_m(R = 5\text{mm}, z)$ for the radial magnetic field component. The harmonics $B_{m}^{nt}(R = 5\text{mm}, z)$ are shown in blue color. The red color corresponds to the harmonics $B_{m}^{dt}(R = 5\text{mm}, z)$ computed by the generalized gradients $C_{m}^{[k]}$ according to Eq. (6.20).

The Taylor expansion of the vector potential in cartesian coordinates through the 6th order can be written as [109]:

$$
A_x = x^2 C_1^{[1]} - \frac{1}{8} x^4 C_1^{[3]} - \frac{1}{8} x^2 y^2 C_1^{[3]} + \frac{1}{192} x^6 C_1^{[5]} + \frac{1}{96} x^4 y^2 C_1^{[5]} + \frac{1}{192} x^2 y^4 C_1^{[5]} + \frac{1}{3} x^4 C_3^{[1]} - \frac{1}{48} x^6 C_3^{[3]} + \frac{1}{24} x^4 y^2 C_3^{[3]} + \frac{1}{16} x^2 y^4 C_3^{[3]} + \frac{1}{5} x^6 C_3^{[5]} - 2 x^4 y^2 C_3^{[5]} + x^2 y^4 C_3^{[5]}
$$

$$
A_y = xy C_1^{[1]} - \frac{1}{8} x^3 y C_1^{[3]} - \frac{1}{8} xy^3 C_1^{[3]} + \frac{1}{192} x^5 y C_1^{[5]} + \frac{1}{96} x^3 y^3 C_1^{[5]} + \frac{1}{192} xy^5 C_1^{[5]} + \frac{1}{3} x^3 y C_3^{[1]} - \frac{1}{48} x^5 y C_3^{[3]} + \frac{1}{24} x^3 y^3 C_3^{[3]} + \frac{1}{16} x^5 y C_3^{[3]} + \frac{1}{5} x^3 y^3 C_3^{[3]} - 2 x^3 y^3 C_3^{[3]} + xy^5 C_5^{[1]}
$$

$$
A_z = -x C_1 + \frac{3}{8} x^3 C_1^{[2]} + \frac{3}{8} xy^2 C_1^{[2]} - \frac{5}{192} x^5 C_1^{[4]} - \frac{5}{96} x^3 y^2 C_1^{[4]} - \frac{5}{192} x^5 C_1^{[4]} - x^3 C_3 + 3 x y^2 C_3^{[2]} + \frac{5}{48} x^5 C_3^{[2]} - \frac{5}{24} x^3 y^2 C_3^{[2]} - \frac{5}{16} x^5 C_3^{[4]} - x^3 C_5 + 10 x^3 y^2 C_5 - 5 x y^4 C_5
$$

(6.22)
Figure 6.11: Vertical field along the longitudinal axis at \( X=Y=0 \) and \( X=0, Y=5 \) mm (left plot); longitudinal field along the longitudinal axis at \( X=Y=0 \) and \( X=0, Y=5 \) mm (right plot).

Figure 6.12: Vertical (left plot) and longitudinal (right plot) field along the longitudinal axis at \( X=3.5 \) mm, \( Y=3.5 \) mm.

The close correspondence between the fitted field and the numerical field data computed by the magnet modelling code Mermaid is shown in the Figs. 6.11, 6.12, 6.13. The blue points show the numerical field data from the Mermaid code while the red curves present the results of the analytical fit based on the Cartesian representation (6.21). Therefore in the expected range of validity (\(|Y| \leq 5\) mm and \(|X| \leq 60\) mm) the field map is in good agreement with the analytical fit.
Figure 6.13: Vertical magnetic field as a function of horizontal and vertical position at Z=75 mm.

6.6 Symplectic integrator

Choosing a Cartesian coordinate system, with the z-axis oriented along the longitudinal wiggler axis, the general Hamiltonian for a charged particle of mass $m$ and charge $e$ in a magnetic vector potential $\mathbf{A}$ is given by

$$
H = -\sqrt{(1 + \delta)^2 - (p_x - \frac{e}{p_0} A_x)^2 - (p_y - \frac{e}{p_0} A_y)^2 - \frac{e}{p_0} A_z}
$$

where $p_0 = m_0 c \beta \gamma$ is design momentum and $p_{x,y} = P_{x,y}/P_0$ refers to the normalized transverse momenta. The ratio between the particle charge and design momentum $p_0$ is $e/p_0 = 1/B_0 \rho$ where $B_0 \rho$ denotes the magnetic rigidity. Expanding the Hamiltonian in the paraxial approximation $(p_i - \frac{e}{p_0} A_i)^2 \ll 1$, it can be simplified to

$$
H \approx -(1 + \delta) + \frac{p_x^2 + p_y^2}{2(1 + \delta)} + \left( \frac{e}{p_0} \right)^2 A_x^2 + A_y^2 - \frac{e}{p_0} A_z x - \frac{e}{p_0} A_z y
$$

The third and fifth terms of equation (6.23) give a transverse momentum kick. The fourth term involves coupling between the momenta and the co-ordinates and the first two terms just generate a drift (a region without magnetic field).

Analytical expressions for the transverse kicks enable us to estimate numerically the influence of the various terms in the field multipole expansion on the particle dynamics.

The motion of the on-energy reference particle entering the wiggler without any orbit deviation is described by the Hamiltonian $H = -\sqrt{1 - (p_x^2 - \frac{e}{p_0} A_x(x_r, 0, z))^2 - \frac{e}{p_0} A_z(x_r, 0, z)}$. Through the wiggler, the particle moves on the reference orbit $x_r$ that is confined within the $y = 0$ plane (where $A_y(x, y = 0, z) = 0$). The reference orbit determined by the on-axis wiggler field (equal to the generalized gradient $C_1$) is given by

$$
x_r(z) = -\frac{e}{p_0} \int_0^z \int_0^{z'} C_1(z'')dz'' , \quad p_{x,z}^r(z) = \frac{dx_r}{dz} = -\frac{e}{p_0} \int_0^z C_1(z')dz'
$$

The reference orbit through one 10 cm wiggler period at the peak field of 1.7 T is shown in Fig. 6.14. Determining the particle dynamics with respect to the reference orbit $x_r(z)$, new canonical variables are defined as the deviation of particle coordinates from the reference orbit via $X = x - x_r, Y = y$.
with the transverse momenta $P_x = p_x - p_x^r$ and $P_y = p_y$. In the case of an on-energy particle the Hamiltonian is written as

$$H = - \left[ 1 - \left( P_x + p_x^r - \frac{e}{p_0} A_x^{\text{dev}} \right)^2 - \left( P_y - \frac{e}{p_0} A_y^{\text{dev}} \right)^2 \right]^{\frac{1}{2}} - \frac{e}{p_0} A_z^{\text{dev}} - \dot{x}^{r} P_x + \dot{p}_x^{r} X \quad (6.24)$$

where the vector potential $A^{\text{dev}} = A(X + x^r, y, z)$. The Hamiltonian equations of motion are the following:

$$x' = \frac{\partial H}{\partial p_x}, \quad y' = \frac{\partial H}{\partial p_y}, \quad p_x' = - \frac{\partial H}{\partial x}, \quad p_y' = - \frac{\partial H}{\partial y} \quad (6.25)$$

where the primes denote $\frac{d}{dt}$.

### 6.6.1 Horizontal kick

Assuming that on-momentum particle enters a wiggler period in the $y = 0$ plane with a horizontal offset $X$, the horizontal kick produced by the wiggler field can easily be evaluated analytically. We also assume that the horizontal displacement $X$ with respect to the reference orbit remains constant inside the wiggler. It is a good approximation in practice.

The vector potential components $A_{x,y,z}^{\text{dev}}$ up to 4th order will be considered for the estimation of kick. The expression $(X + x^r)^4$ is expanded into the series $X^4 + 4X^3(x^r) + 6X^2(x^r)^2 + 4X(x^r)^3 + (x^r)^4$. We will use only the first and second terms from this expansion. Because the amplitude of the reference orbit $x_r$ through the HPM NdFeB wiggler is quite small due to the very small wiggler period, for each multipole component only the term which is proportional to $x^r(z)$ is significant.

Inserting the vector potential components $A_x(X + x^r, 0, z)$, $A_y(X + x^r, 0, z)$ and $A_z(X + x^r, 0, z)$ from Eq. (6.22) into the Hamiltonian (6.24) and using the equations of motion (6.25), we obtain the horizontal kick $\Delta p_x$ produced at the exit of one wiggler period;

$$\Delta p_x = - \int_0^{x_r} \frac{\partial H}{\partial x} = X(\Delta p_x)_1 + X^3(\Delta p_x)_3 \quad (6.26)$$

where

$$(\Delta p_x)_1 = \frac{1}{4} \frac{e}{p_0} \int_0^{x_r} x^r(z) C_1^0(z) dz - 6 \frac{e}{p_0} \int_0^{x_r} x^r(z) C_3(z) dz$$

Figure 6.14: Reference orbit for the one wiggler period.
\[
(\Delta p_x)_3 = -\frac{e}{p_0} \int_0^\infty x^r(z) \left( C_1^4(z) \frac{C_3^3(z)}{48} - \frac{3C_3^2(z)}{4} + 20C_5(z) \right) dz
\]

The term \( \frac{1}{4} \frac{e}{p_0} \int_0^\infty x^r(z) C_1^2(z) dz \) is always present. The second term \(-6\frac{e}{p_0} \int_0^\infty x^r(z) C_3(z) dz\) is driven by the azimuthal sextupole field component. The resulting sign of \((\Delta p_x)_1\) (linear part of the horizontal kick) depends on the relation between \(C_1\) and \(C_3\). The higher order azimuthal harmonics also contribute to the linear focusing but their relative contributions can be neglected because they are of order \((x^r)^2\) and higher. In the limit of an infinitely wide wiggler i.e. one with a pole width much larger than the wiggler period, the resulting horizontal focusing \(\int x^r(z)[C_1^2(z)/4 - 6C_3(z)]dz\) goes to zero.

Integrating the equations for the \(\Delta p_x\), the numerical values of the kicks for each individual term are presented in the Table 6.5. The dependence of the resulting total horizontal kick on the horizontal displacement \(X\) is shown in Fig. 6.15 (left plot).

![Graph showing horizontal kick produced by one wiggler period in the planes \(Y = \pm 6.0\) mm (left plot); vertical kick through one wiggler period in the planes \(X = \pm 30.0\) mm (right plot).](image)

**Figure 6.15:** Horizontal kick produced by one wiggler period in the planes \(Y = \pm 6.0\) mm (left plot); vertical kick through one wiggler period in the planes \(X = \pm 30.0\) mm (right plot).

| \((\Delta p_x)_1\) | \(\frac{1}{4} \frac{e}{p_0} \int_0^\infty x^r(z) C_1^2(z) dz\) | \(-0.000599375\ m^{-1}\) |
| \((\Delta p_x)_2\) | \(-6\frac{e}{p_0} \int_0^\infty x^r(z) C_3(z) dz\) | \(0.00058896\ m^{-1}\) |
| \((\Delta p_x)_3\) | \(-\frac{e}{p_0} \int_0^\infty x^r(z) \frac{C_1^2(z)}{48} dz\) | \(-0.20578\ m^{-3}\) |
| \((\Delta p_x)_4\) | \(\frac{e}{p_0} \int_0^\infty x^r(z) \frac{3C_3^2(z)}{4} dz\) | \(0.302758\ m^{-3}\) |
| \((\Delta p_x)_5\) | \(-\frac{e}{p_0} \int_0^\infty x^r(z) 20C_5(z) dz\) | \(0.904057\ m^{-3}\) |
| \((\Delta p_x)_6\) | \((\Delta p_x)_{1a} + (\Delta p_x)_{1b}\) | \(0.00001\ m^{-1}\) |
| \((\Delta p_x)_7\) | \((\Delta p_x)_{3a} + (\Delta p_x)_{3b} + (\Delta p_x)_{3c}\) | \(1.0011\ m^{-3}\) |
6.6.2 Vertical kick

The same approach can be used to evaluate the vertical kick produced by one wiggler period. Assuming that the initial conditions of on-momentum particles at the entrance of the wiggler are only in the \(x = 0\) plane and that the vertical displacement \(Y\) is constant with respect to the reference orbit, the vertical kick through 4\(^{th}\) order in \(y\) and first order in the reference orbit displacement \(x'(z)\) is expressed by

\[
\Delta p_y = - \int_0^{\lambda_w} \frac{\partial H}{\partial y} = Y(\Delta p_y)_1 + Y^3(\Delta p_y)_3
\]

where

\[
(\Delta p_y)_1 = \frac{3}{4} \frac{e}{p_0} \int_0^{\lambda_w} x'(z)C_1^{[2]}(z)dz + 6 \frac{e}{p_0} \int_0^{\lambda_w} x'(z)C_3(z)dz
\]

\[
(\Delta p_y)_3 = - \frac{e}{p_0} \int_0^{\lambda_w} x'(z) \left( \frac{5C_1^{[4]}(z)}{48} + \frac{5C_3^{[2]}(z)}{4} + 20C_5 \right)dz
\]

From Tables 6.5 and 6.6 one can see that the wiggler naturally provides vertical focusing and almost no focusing horizontally. The wiggler nonlinearities do not have the same form as those of a standard octupole magnet, e.g., \((\Delta p_y)_3 \neq (\Delta p_x)_3\). This is the reason why it is difficult to approximate the wiggler nonlinearities by standard multipoles. The vertical kick produced at the exit of one wiggler period is shown in Fig. 6.15 (right plot).

Table 6.6: Terms of the vertical kick produced by one period of HPM NdFeB wiggler

| \((\Delta p_y)_1\)   | \(\frac{3}{4} \frac{e}{p_0} \int_0^{\lambda_w} x'(z)C_1^{[2]}(z)dz\) | \(-0.00179813 \text{ m}^{-1}\) |
| \((\Delta p_y)_1\)   | \(\frac{6}{p_0} \int_0^{\lambda_w} x'(z)C_3(z)dz\) | \(0.00058896 \text{ m}^{-1}\) |
| \((\Delta p_y)_3\)   | \(- \frac{e}{p_0} \int_0^{\lambda_w} x'(z) \frac{5C_1^{[4]}(z)}{48}dz\) | \(-1.02854 \text{ m}^{-3}\) |
| \((\Delta p_y)_3\)   | \(- \frac{e}{p_0} \int_0^{\lambda_w} x'(z) \frac{5C_3^{[2]}(z)}{4}dz\) | \(-0.504597 \text{ m}^{-3}\) |
| \((\Delta p_y)_3\)   | \(- \frac{e}{p_0} \int_0^{\lambda_w} x'(z)20C_5(z)dz\) | \(0.904057 \text{ m}^{-3}\) |
| \((\Delta p_y)_1\)   | \((\Delta p_y)_1 + (\Delta p_y)_1\) | \(-0.002387 \text{ m}^{-1}\) |
| \((\Delta p_y)_3\)   | \((\Delta p_y)_3 + (\Delta p_y)_3 + (\Delta p_y)_3\) | \(-0.62908 \text{ m}^{-3}\) |

6.7 Dynamic aperture in presence of wiggler nonlinearities

Mapped insertion devices definition in BETA-LNS code are based on the description of insertion devices by interpolation tables which provide the kicks \(x'\) and \(y'\) as a function of the coordinates
$x$, $y$ of the particle passing the element. This description is more general and can be more precise than the analytical description by a mathematical formula.

Equations (6.26) and (6.27) were used to calculate interpolation tables for one NdFeB wiggler module which consists of 20 periods. The dynamic aperture in presence of wiggler nonlinearities which were introduced in the form of interpolation tables was checked by BETA-LNS code.

The use of mapped insertion devices does not lead to a reduction of the dynamic aperture when the sextupoles are turned on. With nonlinearities induced only by wigglers (if the sextupoles are turned off), the dynamic aperture is much larger than the physical aperture $12 \text{ mm} \times 60 \text{ mm}$ of the vacuum chamber in the straight sections.
Chapter 7

Tolerances for alignment errors and correction of vertical dispersion and betatron coupling

7.1 Alignment errors

7.1.1 Error sources

The lattice design of the CLIC damping ring presented before was based on the ideal lattice. The next step is to test the lattice performance for various error sources. Imperfections related to alignment errors of the elements along the ring and field errors always occur in real machines. Horizontal, vertical and longitudinal displacements, small roll angles and field errors of magnet elements excite vertical and horizontal orbit distortions, vertical dispersion and betatron coupling. Misalignments can be assumed as randomly distributed over the ring. They may be partially correlated, if magnet groups are mounted on girders.

Multipolar errors in magnets do not distort the orbit but they can excite higher order resonances that may impact the dynamic aperture. Multipolar errors are usually systematic deviations from the ideal fields and they characterize the quality of the magnets.

For the ideal lattice the closed orbit is zero everywhere for a particle with design momentum, and it is defined by the dispersion $D(s)$ if the relative momentum deviation $\delta$ is not zero. In the real lattice, transverse kicks $\Delta x', \Delta y'$ caused by transverse alignment errors $\Delta X, \Delta Y$ excite a nonzero closed orbit. If the dipolar kicks are too strong it may happen that no closed orbit exists. Passing through the magnetic field the particle experiences the horizontal and vertical kicks which are given by an integration of the Lorentz force over the length of the magnet $L$.

\[
\Delta x' = -\frac{1}{B\rho} \int_0^L B_y dl, \quad \Delta y' = \frac{1}{B\rho} \int_0^L B_x dl
\]

The integrated kicks from a quadrupole with transverse displacements $\Delta X$ and $\Delta Y$ (offsets of the beam from the quadrupole center) are

\[
\Delta x' = -K_1 L \Delta X, \quad \Delta y' = K_1 L \Delta Y.
\] (7.1)
The kicks from a sextupole are given by
\[ \Delta x' = -\frac{K_2}{2} L(\Delta X^2 - \Delta Y^2), \quad \Delta y' = K_2 L \Delta X \Delta Y, \tag{7.2} \]
where \( L \) is the length of magnet. \( K_1, K_2 \) are the normalized quadrupole and sextupole field, respectively:
\[ K_1 = \frac{e}{p_0} \frac{\partial B_y}{\partial x}, \quad K_2 = \frac{e}{p_0} \frac{\partial^2 B_y}{\partial^2 x} \tag{7.3} \]
As one can see, vertical displacements of sextupoles generate coupling between the transverse planes, whereas a displaced quadrupole does not introduce any coupling. The contribution from the sextupoles to the closed orbit distortion is small due to the quadratic dependence on \( \Delta X \) and \( \Delta Y \).

The kicks from a small roll angle \( \Theta \) of a dipole magnet are given by
\[ \Delta x' = -\frac{1}{B \rho} \int B \Theta^2 \frac{\partial}{\partial x} dl, \quad \Delta y' = \frac{1}{B \rho} \int B \Theta dl. \tag{7.4} \]
The vertical kick coming from the roll angle of the dipole magnet is larger than the horizontal kick. Quadrupole roll angles and gradient field errors induce beta-beat and betatron tune shift
\[ \frac{\Delta \beta(s)}{\beta(s)} = \frac{1}{2 \sin(2\pi \nu)} \sum_{i=1}^{N} \beta(s_i)[(K_1(s_i) + \Delta K_1(s_i))(\cos 2\Theta(s_i) - 1) + \Delta K_1(s_i)] \int_{C} \beta(s)(s) ds \tag{7.5} \]
\[ \Delta \nu = \nu - \nu_0 = \frac{1}{2 \sin(2\pi \nu_0)} \int_{C} \beta(s)(s) ds \tag{7.6} \]
where \( N \) is the number of quadrupoles in the ring. \( \Delta K_1(s_i) \) denotes the field error of the \( i \)th quadrupole located at \( s_i \). A quadrupole roll angle \( \Theta(s_i) \) introduces a gradient field error \( \Delta K_1(s_i) \approx -K_1(s_i)(\Theta(s_i))^2/2 \) and, more importantly, it gives rise to betatron coupling in proportion to \( \Theta(s_i) \).

There are many kicks from all magnet misalignments. Assuming that the error distribution is Gaussian, the orbit distortion at the lattice element \( k \) can be expressed by:
\[ x_k = \frac{\sqrt{\beta_{zk}}}{2 \sin(\pi \nu_{zk})} \sum_{i=1}^{N} \sqrt{\beta_{xi}} \Delta x_i' \cos(|\phi_{zk} - \phi_{zi}| - \pi \nu_{zk}) \tag{7.7} \]
A tracking code allowing element misalignments to be set and containing a closed orbit finder is usually applied. Errors are set by a random generator. Many different “seeds” of random numbers have to be tried and averaged in order to obtain a statistically significant result.

### 7.1.2 Equation of motion

Neglecting the effects of synchrotron radiation, the transverse particle motion can be written \cite{111}
\[ x'' + (1 - \Delta) \left[(K_1 + G_{xc}^2)x + K_{1y}y + \frac{K_2}{2}(x^2 - y^2)\right] = G + (1 - \Delta)G_{xc} \]
\[ y'' - (1 - \Delta) \left[K_{1y}y + K_{1x}x + K_{2xy}y\right] = (1 - \Delta)G_{yc}. \tag{7.8} \]
Here, \( \Delta = (p - p_0)/p \) where \( p_0 \) is the reference momentum, \( G(s) = 1/\rho(s) \) the inverse bending radius, and \( G_{xc}, G_{yc} \) are the inverse bending radii of the horizontal and vertical dipole correctors.
$K_1$ and $K_2$ are normalized quadrupole and sextupole fields given by Eq. (7.3). $\tilde{K}_1$ is the normalized skew field given by

$$\tilde{K}_1 = \frac{e}{p_0} \frac{\partial B_x}{\partial x}.$$  

Separating the motion into three portions, namely the periodic closed orbit $x_c$, the first order energy dependence of the closed orbit $D_x \delta$ and the betatron motion $x_\beta$, the particle transverse co-ordinates can be expressed by

$$x = x_c + x_\beta + D_x \delta,$$

$$y = y_c + y_\beta + D_y \delta,$$

where $\delta$ is the relative energy deviation. From Eq. (7.8), the closed orbit is found as

$$x''_c + (K_1 + G_2^2)x_c + \tilde{K}_1 y_c + \frac{K_2}{2}(x_c^2 - y_c^2) = G_{xc},$$

$$y''_c - K_1 y_c + \tilde{K}_1 x_c - K_2 x_c y_c = G_{yc}. \quad (7.9)$$

The equations for the dispersion functions and betatron motion are

$$D_x'' + (K_1 + G_2^2)D_x + \tilde{K}_1 D_y + K_2(x_c x_y - y_c y_x) = G_x,$$

$$G - G_{xc} + (K_1 + G_2^2)x_c + \tilde{K}_1 y_c + \frac{K_2}{2}(x_c^2 - y_c^2),$$

$$D_y'' - K_1 D_y + \tilde{K}_1 D_x - K_2(x_c D_y + y_c D_x) =$$

$$-G_{yc} - K_1 y_c + \tilde{K}_1 x_c - K_2 x_c y_c, \quad (7.10)$$

and

$$x''_\beta + (K_1 + G_2^2)x_\beta + \tilde{K}_1 y_\beta + K_2(x_c x_\beta - y_c y_\beta) = 0,$$

$$y''_\beta - K_1 y_\beta + \tilde{K}_1 x_\beta - K_2(y_c x_\beta + x_c y_\beta) = 0. \quad (7.11)$$

To simplify equations (7.9), (7.10), (7.11) the following assumptions (limit of weak coupling) will be made in the next sections:

- The horizontal dispersion is larger than the vertical dispersion, $D_x \gg D_y$.

- Weak coupling approximation, i.e., the horizontal emittance is much larger than the vertical emittance, $x_\beta \gg y_\beta$.

Magnetic errors or off-axis orbit in sextupoles can generate a significant vertical emittance by:

- transferring horizontal betatron motion into the vertical plane; this is called betatron coupling,

- generating vertical dispersion or transferring horizontal dispersion into the vertical plane.

As one can see from Eqs. (7.9), (7.10), and (7.11), it is easy to distinguish 3 types of sources which increase the vertical emittance via the betatron coupling $C_\beta$ or the vertical dispersion $D_y$:

1. Transverse quadrupole misalignments, dipole errors and their effects:

- Dipolar tilt errors $\rightarrow D_y$
• Dipolar orbit correctors $\rightarrow D_y$
• Vertical closed orbit (CO) in quadrupoles $\rightarrow D_y$

2. Sextupole misalignments and their effects:
• Vertical sextupole displacements $\rightarrow D_y, C_\beta$
• Vertical CO in sextupoles $\rightarrow D_y, C_\beta$

3. Quadrupole tilt errors:
• Skew quadrupoles $\rightarrow D_y, C_\beta$

A vertical dipole field and a non-zero vertical orbit in the quadrupole magnets will introduce some vertical dispersion. Second, a non-zero vertical orbit through the sextupole magnets, vertical sextupole misalignments, or rotational misalignments of the quadrupoles couple the particle motion in the horizontal and vertical planes. In the next section, the transverse misalignments of quadrupoles and sextupoles are considered. In addition, the effect of random tilt errors of the quadrupoles and the bending dipoles are considered also.

### 7.2 Vertical emittance increase due to random errors

#### 7.2.1 The contribution of the vertical dispersion to the vertical emittance

A vertical dispersion results from alignment errors and a non-zero closed orbit. In the limit of weak coupling, Eq. (7.10) for the vertical dispersion can be simplified to

$$D_y' - K_1 D_y \simeq -G_{yc} - K_1 y_c - \tilde{K}_1 D_x + K_2 y_c D_x$$

Using the periodic Green function, the solution for $D_y$ is found by

$$D_y = \frac{\sqrt{\beta_y(s)}}{2 \sin \pi \nu_y} \int_s^{s+C} \sqrt{\beta_y(z)} \cos[\phi_y(s) - \phi_y(z) + \pi \nu_y] F(z) dz$$

(7.12)

where

$$F(z) = (K_2 D_x - K_1)y_c - \tilde{K}_1 D_x - G_{yc}.$$  

(7.13)

It is important to notice that the term $(K_2 D_x - K_1)$ is proportional to the local chromaticity since the chromaticity is defined as

$$\xi_y = \frac{d\nu_y}{dp/p_0} = \frac{1}{4\pi} \int (K_1 - K_2 D_x) \beta_y ds$$

The local chromatic correction reduces the driving term $F(z)$, which in turn reduces the vertical dispersion. In the present design of the CLIC damping ring, the chromaticity correction is global. It means that chromaticity is compensated by the sextupoles which are located only in the arcs. Thus, the average chromaticity is zero, but the local chromaticity is positive in the regions of dispersion to compensate the negative values in the dispersion free regions. While the average chromaticity is zero, the local values are not zero. Therefore, without correction, a vertical closed-orbit distortion can generate large vertical dispersion.
The vertical dispersion leads to a coupling between the vertical phase space and the energy deviation induced by the synchrotron radiation. The Courant-Snyder dispersion invariant \( H_y \) defines the fundamental increase in the vertical phase space volume. The dispersion invariant \( H_y \) is given by

\[
H_y(s) = \frac{1}{4\beta_y \sin^2 \pi \nu_y} \left| \int s+C F(z) e^{i\psi_y(s) - \psi_y(z) + \pi \nu_y} \, dz \right|^2 \approx 2 \frac{D_y^2(s)}{\beta_y} \quad (7.14)
\]

where, \( \alpha_y, \beta_y \) and \( \gamma_y \) are the Twiss lattice parameters. The vertical emittance increase due to the vertical dispersion occurs because the noise due to the synchrotron radiation can couple into the vertical plane when the dispsersion is non-zero, or, more generally, the eigenvectors of the 1-turn transport matrix are rotated in the bending magnets. Thus, this effect leads to a growth of the vertical emittance. Here, the effect is not local. It depends, e.g., on the dispersion generated by previous bending magnets. The increase in the beam emittance must be reduced by correcting the sources of the coupling in the damping ring. The contribution to the vertical zero-current emittance (no effect of IBS, in future, vertical and horizontal zero-current emittances will be denoted as \( \epsilon_{y0} \) and \( \epsilon_{x0} \)) from the vertical dispersion is given by

\[
\epsilon_{y0,d} = C q_0^2 \frac{J_y}{J_y} \left( \frac{G}{G^2} \frac{\langle D_y^2 \rangle}{\beta_y} \right) \approx 2 J_e \langle D_y^2 \rangle \beta_y \sigma_p^2. \quad (7.15)
\]

Considering the case when the orbit is already corrected by dipole correctors, we can estimate the square of the vertical dispersion generated by an ensemble of random errors with a Gaussian distribution, e.g.,

- by uncorrelated sextupole misalignments [115] \( Y_s \):

\[
\langle D_y^2 \rangle_{\text{six misalign}} = \frac{1}{8 \sin^2 \pi \nu_y} \sum_{\text{six}} (K_2 L)^2 Y_s^2 \beta_y D_x^2,
\]

- by uncorrelated quadrupole rotational errors \( \Theta_q \):

\[
\langle D_y^2 \rangle_{\text{quad rotation}} = \frac{1}{2 \sin^2 \pi \nu_y} \sum_{\text{quad}} (K_1 L)^2 \Theta_q^2 \beta_y D_x^2,
\]

- or by the dipole kicks:

\[
\langle D_y^2 \rangle_{\text{dipole kicks}} = \frac{\langle y_2^2 \rangle}{\beta_y} = \frac{1}{8 \sin^2 \pi \nu_y} \sum_{\text{kicks}} (GL)^2 \beta_y
\]

where \( G(s) = G_{yc} + G \Theta_B + K_1 Y_s \) is the function of the vertical dipole kicks, the angle \( \Theta_B \) is the rotational errors of the bending magnets and \( Y_s \) is the rms vertical misalignment of the sextupoles with respect to the closed orbit.

The first and second expression stated above do not depend upon the closed orbit, while the third expression is defined by the square of the residual for a corrected orbit. It means that the function \( (G(s))^2 \) needs to be minimized by the efficient choice of the dipole kicks \( G_{yc} \) for the correctors.
7.2.2 The contribution of the betatron coupling to the vertical emittance

In the limit of the weak coupling, Eq. (7.11) for the vertical betatron motion is simplified to:

\[ y''_β - K_1 y_β = (K_2 y_c - K_1) x_β \]

More precisely, in addition to betatron oscillations described by this equation, the motion is damped due to radiation damping. The betatron coupling couples the vertical emittance to the synchrotron radiation, which excites the horizontal plane through the horizontal dispersion. Far from linear coupling resonances \( \nu_x \pm \nu_y = n \) and when the damping per turn is small compared to the sum and difference of the two transverse tunes \( 2\pi(\nu_x \pm \nu_y) \gg \alpha_x T_0, \alpha_y T_0 \), the increase of the vertical zero-current emittance due to weak betatron coupling can be expressed as [113]:

\[
\epsilon_{y0,β} =\frac{C_q γ^2}{16\mathcal{J}_y \sqrt{G^2}} \int C H_x [G^3] \left[ \sum \pm \frac{|Q_±(s)|^2}{\sin^2 \pi Δν_±} + 2\text{Re} \frac{Q_+(s)Q_-(s)}{\sin(\pi Δν_+)\sin(\pi Δν_-)} \right] ds
\]

where \( Q_±(s) = \int_{s}^{s+C} (K_2 y_c - K_1) \sqrt{β_x/β_y} e^{i[(ψ_ν(s)±ψ_y(s))-(ψ_x(z)±ψ_y(z))±2\pi(ν_x±ν_y)]} dz \)

The sum over \( ± \) denotes a sum over both the ” + ” term (sum resonance) and the ” − ” term (difference resonance), \( Δν_+ = ν_x + ν_y \) and \( Δν_- = ν_x - ν_y \). It is to be noted that this equation is not valid near the coupling resonance. The definition of the coupling coefficients \( Q_± \) is an s-dependent generalization of the more common definitions [114], which are found from the Fourier component at the sum and difference resonance.

Random quadrupole rotations and random sextupole misalignments induce not only vertical dispersion, but also betatron coupling that increases the vertical emittance as follows [115]:

- from uncorrelated sextupole misalignments:

\[
\langle ϵ_{y0} \rangle = ϵ_x \frac{α_x}{α_y} \frac{(1 - \cos 2\pi ν_x \cos 2\pi ν_y)}{(\cos 2\pi ν_x - \cos 2\pi ν_y)^2} \sum_{s_{ext}} (K_2 L)^2 Y_s^2 β_x β_y , \quad (7.18)
\]

- from uncorrelated quadrupole rotational errors:

\[
\langle ϵ_{y0} \rangle = ϵ_x \frac{α_x}{α_y} \frac{(1 - \cos 2\pi ν_x \cos 2\pi ν_y)}{(\cos 2\pi ν_x - \cos 2\pi ν_y)^2} \sum_{quad} (K_1 L)^2 Θ_q^2 \beta_x β_y . \quad (7.19)
\]

Therefore, the contribution of the residual orbit \( ⟨y_0^2⟩ \) after correction to the vertical emittance is also a function of the linear coupling. If the vertical orbit is compensated by \( N_{corr} \) dipole correctors, then the contribution to the vertical emittance from the correctors alone can be expressed as [115]:

\[
\langle ϵ_{y0} \rangle = \sum_{Δν_i,ψ_i} \frac{ϵ_x α_x}{32 \sin^2 π Δν_i α_y} \frac{⟨y_0^2⟩_{ψ_i}}{β_y} \sum_{n_c} \left[ \frac{1}{n_c} \int_{n_c}^{n_c+1} K_2(z)β_y(z) \sqrt{β_x(z)} e^{iψ_i} dz \right]^2 . \quad (7.20)
\]

The sum over \( Δν_i \) and \( ψ_i \) is a sum over two values of \( Δν = ν_x - ν_y \), \( Δν = ν_x + ν_y \) and two values of \( ψ \) associated with each value for \( Δν_i \) as:

for \( Δν_1 = ν_x + ν_y \), \( ψ_1 = ψ_x + 2ψ_y \) and \( ψ_2 = ψ_x \)

for \( Δν_2 = ν_x - ν_y \), \( ψ_1 = ψ_x - 2ψ_y \) and \( ψ_2 = ψ_x \).

The integral is calculated between correctors \( n_c \) and \( n_{c+1} \) rather than over the entire ring. Eq. (7.20) is valid only when the closed orbit is broken into short segments by correctors. If the orbit is broken at every sextupole, then Eq. (7.20) reduces to Eq. (7.18). Thus, for comparable residual orbit and sextupole misalignment \( (y_c ≈ Y_s) \) the contribution to the vertical emittance from the orbit should be less than the contribution from the misalignment since the orbit is typically correlated across many sextupoles.
7.3 Estimates for alignment sensitivities of the emittance

In the general case, the sensitivities of the closed orbit and optical functions to systematic misalignments are smaller than those to the random misalignments. This occurs because the phase advance in Eq. (7.14) leads to a cancellation of the contribution. In a simple periodic system, the contribution from the systematic errors has the form [116]:

\[
\text{(systematic)} \sim \frac{1}{\sin^2(\pi \nu/N_s)} \frac{1}{\sin^2 \pi \nu_c}
\]

while \( (\text{random}) \sim \frac{N}{\sin^2 \pi \nu} \).

Here, \( \nu \) is the tune: \( \nu_y, \nu_x \pm \nu_y \), \( N_s \) is the number of superperiods, \( N \) is the number of cells, and \( \nu_c \) is the relevant phase advance per cell: \( \nu_c = \nu_{yc} \), or \( \nu_c = \nu_{xc} \pm \nu_{yc} \). Thus, provided that the tune per superperiod is far from resonance and \( \nu_c \gg 1/\sqrt{\pi N} \), the emittance is less sensitive to systematic errors than to random errors; in the CLIC damping ring, \( \nu_{yc} = \pi/2 \) while \( 1/\sqrt{\pi N} = 0.032 \).

A group of magnets is usually aligned to very high precision (<50 \( \mu \)m) to the girder which is a rather stiff piece of steel. The transitions between girders are made at locations of low beta since there the closed orbit is less sensitive. However, in the further studies, we will consider random misalignments only.

Summarizing Eqs. (7.16), (7.17), (7.18) and (7.19) for the vertical emittance, we can make some simple estimates of the sensitivity of the vertical emittance to uncorrelated sextupole misalignments \( \langle Y^2_{\text{sext}} \rangle \) and quadrupole rotations \( \langle \Delta \Theta^2_{\text{quad}} \rangle \), both of which generate vertical dispersion and betatron coupling.

The vertical emittance from uncorrelated sextupole misalignments may be written [117]:

\[
\epsilon_{y0} = \langle Y^2_{\text{sext}} \rangle \left[ \Sigma^K_2 J_x (1 - \cos 2\pi \nu_x \cos 2\pi \nu_y) \epsilon_x + \Sigma^K_2 J_y (\cos 2\pi \nu_x - \cos 2\pi \nu_y) \epsilon_y \right]
\]

and the vertical emittance from uncorrelated quadrupole rotations may be written

\[
\epsilon_{y0} = \langle \Delta \Theta^2_{\text{quad}} \rangle \left[ \Sigma^K_1 J_x (1 - \cos 2\pi \nu_x \cos 2\pi \nu_y) \epsilon_x + \Sigma^K_1 J_y (\cos 2\pi \nu_x - \cos 2\pi \nu_y) \epsilon_y \right]
\]

where the magnet sums over the sextupoles and quadrupoles are given by

\[
\Sigma^K_2 = \sum_{\text{sext}} \beta_x \beta_y (K_2 l)^2,
\]

\[
\Sigma^K_1 = \sum_{\text{quad}} \beta_x \beta_y (K_1 l)^2,
\]

\[
\Sigma^K_2 = \sum_{\text{sext}} \beta_y (K_2 l D_x)^2,
\]

\[
\Sigma^K_1 = \sum_{\text{quad}} \beta_y (K_1 l D_x)^2.
\]

The sensitivity is defined as the rms misalignment that on its own will generate the specified equilibrium vertical emittance. The formulae given above should not be used to estimate the resulting vertical dispersion or vertical emittance, if the closed orbit is uncorrected (and, hence, contains large correlations). We can also write down a simple expression to estimate the closed-orbit distortion in response to an uncorrelated quadrupole misalignment

\[
\left( \frac{\beta_y}{\beta_{\text{co}}} \right) = \langle \Delta Y^2_{\text{quad}} \rangle \frac{\sum_{\text{quad}} \beta_y (K_1 l)^2}{8 \sin^2 \pi \nu_y}.
\]
The tracking code BETA-LNS [80] was used to study the sensitivity of the damping ring lattice to alignment errors. The alignment errors were assigned to the elements by a random generator. Many different ”seeds” of random numbers have been averaged by the BETA-LNS code in order to obtain a statistically significant result. In the simulations, the random errors were generated with Gaussian distributions truncated at \( \pm 3\sigma \).

As one can see from Fig. 7.1a, quadrupole vertical misalignments \( \langle \Delta Y_{\text{quad}} \rangle \) randomly assigned to all quadrupole magnets have a strong impact on the closed orbit in the CLIC damping ring. The simulation was done with sextupoles turned on (switched on) and at \( \langle \Delta Y_{\text{sext}} \rangle = 0 \).

![Figure 7.1: a) Correlation between rms closed orbit distortion and quadrupole vertical misalignment \( \langle \Delta Y_{\text{quad}} \rangle \) at turned on sextupoles with \( \langle \Delta Y_{\text{sext}} \rangle = 0 \) (left plot); b) Correlation between rms vertical closed orbit distortion and rms rotational errors of the bending magnets at the turned on sextupoles and at \( \langle \Delta Y_{\text{quad}} \rangle = \langle \Delta Y_{\text{sext}} \rangle = 0 \) (right plot).](image)

Figure 7.2: a) The zero-current emittance ratio \( \varepsilon_{y0}/\varepsilon_{x0} \) including contribution from the vertical dispersion and betatron coupling as a function of rms quadrupole rotational error \( \langle \Delta \Theta_{\text{quad}} \rangle \). The dashed line shows the fitted quadratic curve \( \varepsilon_{y0}/\varepsilon_{x0} = 3.261 \times 10^{-6}\langle \Delta \Theta_{\text{quad}}^2 \rangle \) (left plot); b) RMS vertical dispersion as a function of the rms quadrupole rotational error \( \langle \Delta \Theta_{\text{quad}} \rangle \) (right plot). The simulations shown on both plots were done with turned on sextupoles at \( \langle \Delta Y_{\text{sext}} \rangle = 0 \).
The random tilt errors $\langle \Delta \Theta_{\text{arc dipole}} \rangle$ of the dipole bending magnets induce quite small closed orbit distortions (compared to the quadrupole misalignment), as shown in Fig. 7.1b. For $\langle \Delta \Theta_{\text{arc dipole}} \rangle = 100 \, \mu\text{rad}$, the vertical and horizontal orbit distortion are $38 \, \mu\text{m}$ and $\sim 0.01 \, \mu\text{m}$, respectively.

The result of the simulations carried out for the CLIC damping ring lattice is shown in Figs. 7.2 and 7.3. They illustrate the sensitivity of the vertical dispersion and zero-current transverse emittance ratio $\epsilon_y/\epsilon_x$ to the uncorrelated quadrupole rotations and sextupole misalignments. The effect from the quadrupole rotations was computed with zero sextupole misalignments, but with sextupoles turned on. Likewise, when the effects from sextupole misalignments were computed, the quadrupole rotations were set to zero.

The blue rhombic points and the points with error bar correspond to the rms dispersion and emittance ratio $\epsilon_y/\epsilon_x$, respectively, as computed by the BETA-LNS code. The dashed lines represent a fit to the data.

As one can see from these simulations, fairly small magnet errors introduce unacceptable distortions in the closed orbit as well as vertical dispersion and coupling, due to the strong focusing optics of the damping ring.

Figure 7.3: a) The zero-current emittance ratio $\epsilon_y/\epsilon_x$ including contribution from the vertical dispersion and betatron coupling as a function of the rms vertical sextupole misalignment $\langle \Delta Y_{\text{sext}} \rangle$. The dashed line shows the fitted quadratic curve $\epsilon_y/\epsilon_x = 1.57 \times 10^{-4} \langle \Delta Y_{\text{sext}}^2 \rangle$ (left plot); b) RMS vertical dispersion as a function of the rms vertical sextupole misalignment $\langle \Delta Y_{\text{sext}} \rangle$ (right plot). The simulations shown on both plots were done at $\langle \Delta Y_{\text{quad}} \rangle = 0$.

### 7.4 Closed orbit correction

#### 7.4.1 Correctors and BPMs

The parasitic dipole kicks due to misalignments are compensated by application of appropriate dedicated kicks. For these additional kicks, either small dipole corrector magnets of variable field or transverse movement of the quadrupole magnets are needed. To control the beam orbit, beam position monitors (BPMs) must be installed at many locations over the ring. Generally it is sufficient to install BPMs and correctors at a quarter betatron wavelength distance. With a phase advance of $\pi/2$, a kick applied at a corrector results in maximum displacement at the subsequent BPM. It is neither possible nor necessary to set BPM/corrector exactly a quarter betatron wavelength apart, but the phase advance has to be smaller than $\pi$. 

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The appropriate locations for BPMs and correctors and the required space for installing the devices has to be taken into account. In order to save space, the correctors can be integrated as additional coils in the quadrupoles or sextupoles. For optimum correction, it is preferable that a corrector is localized close to the source of alignment error. This is an additional advantage of integrated location of correctors.

Since the resolution of conventional BPMs is typically limited to 2-10 μm, the BPMs have to be installed at places where the orbit distortion is maximum. This is important also for the efficiency of dispersion measurements performed by the same BPMs. A slight shift of the RF frequency $\Delta f_{rf}$ causes a small change $\Delta x, \Delta y$ of the closed orbit that can be measured by BPMs. Then the dispersion is calculated as:

$$D_x(s) = -\frac{\Delta x(s)_{rf}}{\Delta f_{rf}} \left( \alpha_p - \frac{1}{\gamma^2} \right), \quad D_y(s) = -\frac{\Delta y(s)_{rf}}{\Delta f_{rf}} \left( \alpha_p - \frac{1}{\gamma^2} \right)$$

As one can see from Eq. (7.7):

- The correctors with vertical dipole field that provide a horizontal kick and BPMs which are selected to detect the horizontal orbit displacement have to be located at places where the $\beta_x$ is maximum.
- The correctors with horizontal dipole field that provide a vertical kick and BPMs which are selected to detect the vertical orbit displacement have to be installed at places where $\beta_y$ is maximum.

In order to meet these requirements and to save space, the correctors providing a horizontal kick (horizontal correctors) will be inserted as additional coils in the focusing quadrupoles. Vertical correctors (which provide a vertical kick by a horizontal dipole field) will be inserted as additional coils in the defocusing quadrupoles. Skew quadrupole corrector may be inserted in some sextupoles of the arcs and in some quadrupoles of the dispersion free straight section.

From a theoretical point of view, the correction of the dispersion at the BPMs does not give the minimum value of the emittance. For this one would have to minimize the dispersion invariant $\mathcal{H}_y$ function in each dipole.

### 7.4.2 Correction strategy

To reach a very low vertical emittance, we need to control the betatron coupling and dispersion. It is necessary to develop an effective correction system which will restore the transverse emittances to the values $\gamma\epsilon_y = 3.4$ nm and $\gamma\epsilon_x = 540$ nm (taking into account IBS) achieved for the ideal machine (without any imperfections). The correction scheme that will be used for the damping ring is the following:

- simultaneous correction of the orbit and the dispersion by dispersion free steering (DFS) method,
- minimization of the vertical dispersion using two or more skew quadrupoles in the arc,
- optimization of the emittance with at least two skew quadrupoles in the straight section.

In order to minimize the vertical betatron coupling it is necessary to locate skew quadrupole correctors in places where the product $\beta_x\beta_y$ is large for maximum efficiency. It is advisable to compensate betatron coupling by skew quad correctors in the dispersion free straight section because the product of $\beta_x\beta_y$ is much larger than in the arcs. An additional advantage of skew quadrupoles correctors installed in the straight section is that they do not generate large value of vertical
dispersion since after closed orbit correction the average value of the horizontal dispersion $D_x$ should be much smaller in the straight sections than in the arcs. Moreover, families of skew quadrupole correctors assigned for compensation of coupling and vertical spurious dispersion, respectively, will be decoupled. To minimize $D_y$ skew quadrupole correctors should be installed in the arc at positions where the horizontal dispersion $D_x$ is largest.

The CLIC damping ring consists of two 48-cell arcs and two wiggler straight sections. The arc cell, shown in Fig. 7.4, comprises one dipole magnet, two identical focusing quadrupoles QF, two identical defocusing quadrupoles QD and three sextupoles SF-SF located between the quadrupoles.

Taking into account the conditions stated above which provide the maximum efficiency for correctors and BPMs, we arranged the horizontal correctors HC as additional coils in the focusing quadrupoles QF, where $\beta_x$ is maximum and the vertical correctors VC are set as additional coils in the SD sextupoles of the arcs, as illustrated in Fig. 7.4. In the dispersion-free FODO straight section, horizontal and vertical correctors are located near each focusing and defocusing quadrupole, respectively. Moreover, three horizontal and vertical correctors are inserted in each dispersion suppressor. Skew quadrupole correctors can be installed in some sextupoles SF. Assuming that the vertical and horizontal beam position can be simultaneously detected by each BPM, we installed two BPMs in each arc cell, as is shown in Fig. 7.4, and also near each quadrupole of the FODO straight section. At the same BPMs the dispersion can be monitored. As a result of this set up, the total number of BPMs are 292 units. The total number of horizontal and vertical correctors over the ring are 246 and 146 units, respectively.

It is to be noted that the choice of location and number of the correctors/BPMs is a tentative one, in order to start the investigation of the closed orbit correction. The necessary number of correctors will be discussed in the next sections where the correction procedure is described.

![Figure 7.4: Preliminary location of the BPMs and correctors over one arc cell.](image)

### 7.4.3 Dispersion free steering

Dispersion free steering [118] consists of a simultaneous correction of the orbit and the dispersion. In most machines the beam position is measured with a set of $N$ beam position monitors (BPMs) which are distributed over the ring. The orbit is corrected with a set of $M$ dipole correctors.
The beam position at the BPMs can be represented by a vector

\[ \mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_N \end{pmatrix}, \]

and the corrector strengths (kicks) by a vector

\[ \mathbf{\theta} = \begin{pmatrix} \theta_1 \\ \theta_2 \\ \vdots \\ \theta_M \end{pmatrix}. \]

A response matrix \( \mathbf{A} \) (dimension \( N \times M \)) is used to describe the relation between corrector kicks and beam position changes at the monitors. The element \( A_{ij} \) of the response matrix corresponds to the orbit shift at the \( i \)th monitor due to a unit kick from the \( j \)th corrector. The elements of the orbit response matrix \( \mathbf{A} \) are determined as:

\[ A_{nm} = \frac{\sqrt{\beta_m \beta_n}}{2 \sin(\pi \nu)} \cos(|\phi_m - \phi_n| - \pi \nu) \] (7.21)

The task of the orbit correction is to find a set of corrector kicks \( \mathbf{\theta} \) that satisfy the following relation:

\[ \mathbf{u} + \mathbf{A} \mathbf{\theta} = 0 \] (7.22)

In general, the number of BPMs \( N \) and the number of correctors \( M \) are not identical and Eq. (7.22) is either over \( (N > M) \) or under \( (N < M) \) constrained. In the former and most frequent case, Eq. (7.22) cannot be solved exactly. Instead, an approximate solution must be found, and commonly used are least square algorithms which minimize the quadratic residual

\[ S = \| \mathbf{u} + \mathbf{A} \mathbf{\theta} \|^2 \] (7.23)

Dispersion free steering is based on the extension of Eq. (7.22) to include the dispersion at the BPMs. The extended linear system is

\[ \left( (1 - \alpha) \mathbf{u} \right) + \left( \alpha \mathbf{D}_u \right) \mathbf{\theta} = 0 \] (7.24)

where vector \( \mathbf{D}_u \) (dimension \( N \)) represents the dispersion at the BPMs. \( \mathbf{B} \) is the \( N \times M \) dispersion response matrix, its elements \( B_{ij} \) giving the dispersion change at the \( i \)th monitor due to a unit kick from the \( j \)th corrector. The weight factor \( \alpha \) is used to shift from a pure orbit \( (\alpha = 0) \) to a pure dispersion correction \( (\alpha = 1) \). In general, the optimum closed orbit and dispersion rms are not of the same magnitude and \( \alpha \) must be adjusted for a given machine. The parameter \( \alpha \) can, in principle, be computed from the BPM accuracy and resolution. Applied to Eq. (7.24), a least square algorithm will minimize

\[ S = (1 - \alpha)^2 \| \mathbf{u} \|^2 + \alpha^2 \| \mathbf{D}_u \|^2 + \| \alpha \mathbf{B} \mathbf{\theta} \|^2 \rightarrow \min \] (7.25)

A fast least-square algorithm [119] (MICADO) is frequently used for orbit correction. It executes an iterative search for the most effective correctors. For the correction by a small number of kicks, MICADO is very efficient. To correct a large number of alignment errors, the SVD method may provide a more effective correction.
Using the BETA-LNS code [80], we have studied dipole correction of the closed orbit and the vertical dispersion which are generated by quadrupole misalignments and rotational errors of the bending magnets. In our simulations the sextupoles are turned on. We assigned tilt errors \( \langle \Delta \Theta_{\text{arc dipole}} \rangle \) of 100 \( \mu \text{rad} \) for all bending magnets and quadrupole misalignment \( \langle \Delta X_{\text{quad}} \rangle = \langle \Delta Y_{\text{quad}} \rangle = 90 \mu \text{m} \) for all quadrupole magnets. The BPMs and dipole correctors are located in the damping ring as discussed in Sec. 7.4.2 (see also Fig. 7.4).

Additional vertical dipole correctors VC which could be inserted in the arc cells, for example in the QF quadrupole, do not improve the effectiveness of the vertical correction since the vertical phase advance per one arc cell is \( \pi/2 \). In total 146 units of vertical correctors VC located in the ring, of which 96 units of the VC correctors are regularly inserted in the arcs. The vertical closed orbit distortion (COD) over half of the ring is shown in Fig. 7.5. The resulting rms value of the vertical COD is 12 \( \mu \text{m} \). Using an increased number of 192 units of VC correctors in the arcs and the same number of VC in the FODO sections, decreases the rms vertical COD to 8 \( \mu \text{m} \).

![Graph of horizontal COD in the arcs](image)

**Figure 7.5:** The vertical closed orbit distortion (COD) over half of the damping ring after the CO correction, for an rms quadrupole misalignment \( \langle \Delta Y_{\text{quad}} \rangle = \langle \Delta X_{\text{quad}} \rangle = 90 \mu \text{m} \), and tilt errors of bending magnets \( \langle \Delta \Theta_{\text{arc dipole}} \rangle = 100 \mu \text{rad} \).

Using only one horizontal corrector HC in the cell (instead of two HC) results in a significant increase of the horizontal orbit in the arcs. Horizontal closed orbits after COD correction for both 96 and 192 units of horizontal correctors HC in the arcs are shown in Figs. 7.6a, 7.6b and 7.7. As one can see from Fig. 7.6b, referring to an rms quadrupole misalignment of 90 \( \mu \text{m} \), the rms horizontal COD after correction is directly proportional to the number of the HC correctors. From Fig. 7.7 it is seen that the residual COD in the FODO straight section is less than a few microns. It is necessary to keep the COD as small as possible in the wiggler FODO sections in order to minimize the transverse emittances and the effect of wiggler nonlinearities. However, if we take into account the BPM resolution, the rms COD in the wiggler sections becomes comparable to the value of the BPM resolution.

The dedicated dipole correction scheme in the arcs, illustrated in Fig. 7.4, provides quite an efficient COD correction that can reduce zero-current vertical emittance \( \gamma_{\epsilon_0} \) down to 2.2 nm for \( \langle \Delta \Theta_{\text{arc dipole}} \rangle = 100 \mu \text{rad} \) and \( \langle \Delta Y_{\text{quad}} \rangle = \langle \Delta X_{\text{quad}} \rangle = 90 \mu \text{m} \), where 77.5 % and 22.5 % of the \( \gamma_{\epsilon_0} \) arise from spurious vertical dispersion and betatron coupling, respectively.
Figure 7.6: a) RMS horizontal closed orbit distortion after correction as a function of the transverse misalignment of the quadrupoles for two different numbers of horizontal dipole correctors HC located in the arcs; b) RMS horizontal closed orbit distortion after correction as a function of the number of correctors HC used in the arcs for the correction, $\langle \Delta Y_{\text{quad}} \rangle = \langle \Delta X_{\text{quad}} \rangle = 90 \, \mu m$.

Figure 7.7: The horizontal COD over half of the damping ring after correction. The upper and lower plot correspond to 96 units and 192 units of the horizontal correctors HC located in the arcs. The rms quadrupole misalignment is $\langle \Delta Y_{\text{quad}} \rangle = \langle \Delta X_{\text{quad}} \rangle = 90 \, \mu m$ for both cases.
The vertical emittance $\gamma\epsilon_{y0}$ is shown in Fig. 7.8 as a function of the quadrupole misalignment at $\langle \Delta\Theta_{\text{arc dipole}} \rangle = 100$ $\mu$rad. For 90 $\mu$m rms quadrupole misalignment, the vertical emittance contribution $\langle \gamma\epsilon_{y0,d} \rangle$ due to the spurious vertical dispersion is equal to 1.7 nm since the average vertical dispersion invariant $\langle H_y \rangle$ is equal to 0.214 $\mu$m.

However, these simulations were done without any misalignments of sextupoles. If we assign transverse rms sextupole misalignments of 20 $\mu$m, then after COD and dispersion correction performed only by dipole correctors, a large vertical emittance $\gamma\epsilon_{y0} = 10.8$ nm remains where 82.7 % and 17.3 % of the $\gamma\epsilon_{y0}$ arise from spurious vertical dispersion and betatron coupling, respectively. The vertical emittance from the rms sextupole misalignments of 20 $\mu$m is larger than the ones due to the residual closed orbit by about a factor of 5. In this case, the contribution $\langle \gamma\epsilon_{y0,d} \rangle$ due to the spurious vertical dispersion and the value of $\langle H_y \rangle$ are equal to 9 nm and 1.13 $\mu$m respectively.

The contributions to the COD produced by kicks from both sextupole and quadrupole elements tend to cancel each other, if the local chromaticity is close to zero. In our case the global (average) chromaticity is zero. However, the local chromaticity in the arcs is positive so as to compensate negative chromaticity in the wiggler straight sections.

Next, we tested the tolerance to the rotational error of quadrupoles of 100 $\mu$rad. After the COD correction at presence of the following errors – $\langle \Delta\Theta_{\text{arc dipole}} \rangle = 100$ $\mu$rad, $\langle \Delta Y_{\text{quad}} \rangle = \langle \Delta X_{\text{quad}} \rangle = 90$ $\mu$m and $\langle \Delta\Theta_{\text{quad}} \rangle = 100$ $\mu$rad –, the resulting vertical emittance is 9 nm, where 86.4 % and 13.6 % of the $\gamma\epsilon_{y0}$ come from spurious vertical dispersion and betatron coupling respectively. To limit the vertical emittance to $\gamma\epsilon_{y} < 3.4$ nm at bunch population of $2.56 \times 10^9$, we need to keep the rms vertical dispersion at a level of less than < 1.5 mm. The vertical dispersion can be corrected with skew quadrupoles in regions of horizontal dispersion. Some skew quadrupole correctors should also be arranged in the wiggler straight section in order to compensate betatron coupling induced by quadrupole rotational errors, vertical misalignment of sextupoles and skew quadrupole correctors inserted in the arcs.

![Figure 7.8: The vertical emittance $\gamma\epsilon_{y0}$ after correction as a function of the transverse rms quadrupole misalignment $\langle \Delta Y_{\text{quad}} \rangle = \langle \Delta X_{\text{quad}} \rangle$ at fixed $\langle \Delta\Theta_{\text{arc dipole}} \rangle = 100$ $\mu$rad. The simulation was done without any misalignments of sextupoles.](image-url)
7.5 Skew quadrupole correction

In the CLIC damping ring, the dominant contribution to the emittance after COD correction is the vertical dispersion [120]. The main contributions to the spurious vertical dispersion are the following ones:

- misalignment of sextupoles,
- vertical COD in the sextupoles,
- tilted quadrupoles.

During correction of the closed-orbit distortion (COD), due to the alignment errors mentioned above, the kicks from the dipole correctors reveal a so-called cross-talk between vertical and horizontal closed orbits [121] (CTCO). The strength of the CTCO is defined by the difference between the two vertical closed orbits, when an horizontal corrector HC is turned on and when it is turned off. In other words, the CTCO is the dependence of the vertical COD on the kicks produced by horizontal correctors HC and the dependence of the horizontal COD on the kicks produced by vertical correctors VC. Minimization of the effect of the CTCO is equivalent to the minimization of the coupling, since the CTCO effect and the vertical betatron motion both result from coupling. If the CTCO could be changed in amplitude without any phase advance modification, the correspondence of rms CTCO to coupling would be strictly regular. Taking into account the analytical formulation presented in Eqs. (7.1), (7.2), (7.21) and (7.22), the orbit cross talk [121] at the $i^{th}$ BPM can be written as:

$$\Delta x_i(\theta^y_j) = \sum_q^{quad} C^x_{iq}(K_1 l_d) q \Theta_q \left[ \Delta Y_q + \sum_j^{vc} C^y_{qj} \theta^y_j \right] +$$

$$\sum_s^{sext} C^x_{is}(K_2 l) s \left[ \Delta Y_s + \sum_j^{vc} C^y_{sj} \theta^y_j \right]^2 - \left[ \Delta X_s + \sum_j^{hc} C^x_{sj} \theta^x_j \right]^2 + \sum_k^{skew cor} C^x_{ik}(\widetilde{K}_1 l_d) k \sum_j C^y_{kj} \theta^y_j$$

(7.26)

and

$$\Delta y_i(\theta^x_j) = \sum_q^{quad} C^y_{iq}(K_1 l_d) q \Theta_q \left[ \Delta X_q + \sum_j^{hc} C^x_{qj} \theta^x_j \right] +$$

$$2 \sum_s^{sext} C^y_{is}(K_2 l) s \left[ \Delta X_s + \sum_j^{hc} C^x_{sj} \theta^x_j \right] \left[ \Delta Y_s + \sum_j^{vc} C^y_{sj} \theta^y_j \right] + \sum_k^{skew cor} C^y_{ik}(\widetilde{K}_1 l_d) k \sum_j C^x_{kj} \theta^x_j$$

(7.27)

Looking at Eq. (7.12), in presence of skew correctors, the spurious vertical dispersion detected by $i^{th}$ BPM can be written by [122]:

$$\Delta D^y_i = \sum_q^{quad} C^y_{iq} \left( (K_1 l D_x) q \Delta \Theta_q - (K_1 l) q \left[ \Delta Y_q + \sum_j^{vc} C^y_{qj} \theta^y_j \right] \right) +$$

$$\sum_s^{sext} C^y_{is}(K_2 l D_x) s \left[ \Delta Y_s + \sum_j^{vc} C^y_{sj} \theta^y_j \right] - \sum_j^{vc} C^y_{ij} \theta^y_j + \sum_k^{skew cor} C^y_{ik}(\widetilde{K}_1 l D_x) k$$

(7.28)
In Eqs. (7.26–7.28) the sensitivity matrix for the orbit or dispersion is

\[ C_{zn}^z = \frac{\sqrt{\beta_{zn}}\beta_{zm}}{2\sin(\pi\nu_z)} \cos(|\phi_{zn} - \phi_{zm}| - \pi\nu_z), \quad z = x, y \] and \( m \& n \in \{i, j, q, s, k\} \)

Here, \( C_{ij}^y \) and \( C_{ij}^x \) are vertical and horizontal response matrix, respectively, given by Eq. (7.21). In Eqs. (7.26–7.27) and (7.28), the following notations are used; \( i \) - BPMs, \( j \) - dipole correctors, \( \theta_{ij}^y \) and \( \theta_{ij}^x \) - vertical and horizontal kicks produced by vertical VC\(_j\) and horizontal HC\(_j\) dipole correctors, \( q \) - quadrupoles, \( s \) - sextupoles, \( k \) - skew quadrupole correctors. We search a set of skew correctors which minimize the CTCOs and \( D_y \) together, namely

\[ \Delta D_y^i \rightarrow 0, \quad \Delta y_i(\theta_{i}^y) \rightarrow 0, \quad \Delta x_i(\theta_{j}^y) \rightarrow 0. \]

Using the BETA-LNS code this procedure of minimization was implemented for the skew quadrupole correction in the CLIC damping ring. The BETA-LNS code enables one to choose a weight factors to the CTCOs, betatron coupling, and \( D_y \) minimization, respectively. The determination of the relative weighting is a matter of choice, but it is obvious that weight factor for the vertical dispersion should be dominant.

We inserted skew quadrupole correctors as additional coils into each second sextupole SD. Thus, 48 units of the skew correctors are included in the damping ring. The standard deviations of random errors assigned for the simulations of the correction are listed in Table 7.1. Note that in these simulations the random errors are generated with Gaussian distributions truncated at ±2σ.

Table 7.1: Random alignment errors assigned to the CLIC damping ring.

<table>
<thead>
<tr>
<th>Imperfections</th>
<th>Simulation</th>
<th>1 r.m.s.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Quadrupole misalignment</td>
<td>( \langle \Delta Y_{\text{quad}} \rangle, \langle \Delta X_{\text{quad}} \rangle )</td>
<td>90 μm.</td>
</tr>
<tr>
<td>Sextupole misalignment</td>
<td>( \langle \Delta Y_{\text{sext}} \rangle, \langle \Delta X_{\text{sext}} \rangle )</td>
<td>40 μm</td>
</tr>
<tr>
<td>Quadrupole rotation</td>
<td>( \langle \Delta \theta_{\text{quad}} \rangle )</td>
<td>100 μrad.</td>
</tr>
<tr>
<td>Dipole rotation</td>
<td>( \langle \Delta \theta_{\text{dipole arc}} \rangle )</td>
<td>100 μrad.</td>
</tr>
<tr>
<td>BPMs resolution</td>
<td>( \langle R_{\text{BPM}} \rangle )</td>
<td>2 μm.</td>
</tr>
</tbody>
</table>

The distributions of zero-current vertical emittance \( \gamma \epsilon_{y0} \) and vertical dispersion invariant \( \mathcal{H}_y \) obtained after the correction of the COD, CTCOs, residual vertical dispersion, and betatron coupling, - carried out by 246 horizontal and 146 vertical dipolar correctors as well as 48 skew quadrupole correctors, - are shown in Figs 7.9 and 7.10, respectively. The distribution of the \( \gamma \epsilon_{y0} \) along half of the ring structure was simulated for 35 different samples of error distributions along the ring. The vertical emittance \( \gamma \epsilon_{y0} \) includes three distinct contributions; a local term (step function with a jump at each error location) and two global components related to betatron coupling and vertical dispersion. The mean value and rms standard deviation of the \( \langle \gamma \epsilon_{y0} \rangle \) are 2.14 nm and ±0.93 nm, respectively. The mean value of the vertical dispersion invariant \( \langle \mathcal{H}_y \rangle \) is 0.248 μm, and its rms standard deviation ±0.114 μm. This means that the contribution of spurious vertical dispersion to the vertical emittance \( \gamma \epsilon_{y0,d} \) is equal to 1.97 ± 0.9 nm (see Eq. 7.15). The calculation was done for the NdFeB hybrid permanent magnet wigglers presented in Sec. 6.2 (\( \lambda_w = 10 \) cm, \( B_w = 1.7 \) T).

The vertical emittance \( \epsilon_{y0} \) after correction is shared as

- contribution of betatron coupling \( \epsilon_{y0,\beta} : 8\% \) of \( \epsilon_{y0} \)
- contribution of spurious vertical dispersion \( \epsilon_{y0,d} : 92\% \) of \( \epsilon_{y0} \)
Figure 7.9: The deviation of the zero-current vertical emittance $\epsilon_{y0}$ along half of the ring after the closed orbit and skew quadrupole correction. The correction was simulated for 35 different sets (samples) of random errors. The blue solid line corresponds to the mean value. The blue dashed lines confine a range of one rms standard deviation around the mean.

Figure 7.10: The deviation of the vertical dispersion invariant $\mathcal{H}_y$ along half of the ring after the closed orbit and skew quadrupole correction. The correction was simulated for 18 different samples of random errors. The red solid line corresponds to the mean value. The red dashed lines confine a range of one rms standard deviation around the mean.
Figure 7.11: Distribution of the horizontal (plot a) and vertical (plot b) closed orbits along half of the ring after the closed orbit and skew quadrupole corrections, which were simulated for 12 different sets (samples) of random errors.

Figure 7.12: Distribution of the horizontal dispersion (plot a) and vertical spurious dispersion (plot b) along half of the ring after the closed orbit and skew quadrupole corrections, which were simulated for 12 different sets (samples) of random errors.
The mean value from betatron coupling \( \langle \epsilon_{y0}/\epsilon_x \rangle \) is 0.13%. The contributions from the vertical dispersion are roughly 10 times larger than the contributions due to the betatron coupling.

In theory, using two skew quadrupole correctors, separated in phase by \( \pi/2 \), one could reduce the expected value of the emittance by a factor proportional to the resonant denominator \( \frac{\pi}{2} \sin^2 \pi \nu_g \). Additional correctors should provide further reductions. A few additional skew correctors may be installed in the FODO sections to control the betatron coupling but as we can see the contribution to the vertical emittance from the betatron coupling is already quite small. Simulations have shown that 96 (instead of 48) skew correctors in the arc (one skew corrector per arc cell) do not significantly reduce the vertical dispersion invariant. Thus, we choose the correction scheme where only 48 skew correctors are located in the arc. The rms strength of the skew and dipole correctors needed for the correction is summarized below:

- r.m.s. kick of the horizontal dipole corrector: \( 0.25 \times 10^{-3} \) rad
- r.m.s. kick of the vertical dipole corrector: \( 0.17 \times 10^{-3} \) rad
- r.m.s. strength of the skew quadrupole corrector: \( 0.21 \) m⁻²

The distributions of the horizontal and vertical COD after the skew quadrupole correction are shown in Figs. 7.11a and 7.11b. The rms value of the orbit deviation does not exceed 18 \( \mu m \) in the arcs and 4 \( \mu m \) in the straight FODO sections. The resolution of the BPMs installed in the arcs has to be as high as possible (and better than 3 \( \mu m \)). The distributions of the horizontal and vertical spurious dispersion are plotted in Figs. 7.12a and 7.12b. The rms value of the vertical dispersion is less than 1.5 mm. As can be seen from Fig. 7.12a, the deviation of \( D_x \) in the arc can be neglected. It changes the zero-current horizontal emittance by less than \( \Delta \gamma \epsilon_x = \pm 8 \) nm. The use of girders could reduce the closed orbit distortion substantially, as was mentioned in the beginning of Section 7.3.

For the design bunch population \( N_{bp} \) of \( 2.56 \times 10^9 \), we must include in our simulation the effect of intra-beam scattering (IBS). The vertical beam size is dominated by vertical dispersion. We performed IBS calculations based on modified Piwinski theory (Sec. 3.5) where we take the mean values \( \langle H_y \rangle = 0.248 \mu m, \langle C_y \rangle = 0.13 \% \), \( \langle \gamma \epsilon_x \rangle = 2.14 \) nm, \( \langle \epsilon_y \rangle = 131 \) nm, \( \sigma_{p0} = 9.15 \times 10^{-4} \) and \( \sigma_{s0} = 1.21 \) mm. The average IBS emittance growth rates over the ring obtained by calculations for these parameters are \( \langle 1/T_x \rangle = 255 \) s⁻¹ (\( \langle T_x \rangle = 3.92 \) ms), \( \langle 1/T_p \rangle = 119 \) s⁻¹ (\( \langle T_p \rangle = 8.4 \) ms) and \( \langle 1/T_y \rangle = 175 \) s⁻¹ (\( \langle T_y \rangle = 5.7 \) ms) at \( N_{bp} = 2.56 \times 10^9 \) and for RF voltage of 2250 kV. Taking into account IBS, the equilibrium emittances, rms bunch length and rms energy spread after the dipole and skew quadrupole correction are the following:

- Horizontal emittance \( \langle \gamma \epsilon_x \rangle \): 530 nm
- Vertical emittance \( \langle \gamma \epsilon_y \rangle \): 3.3 nm
- Emittance ratio \( \epsilon_y/\epsilon_x \): 0.62 %
- RMS bunch length \( \langle \sigma_s \rangle \): 1.63 mm
- RMS energy spread \( \langle \sigma_p \rangle \): 12.35 \( \times 10^{-4} \)
- Longitudinal emittance \( \langle \gamma m_{e} c^2 \sigma_p \sigma_s \rangle \): 4892 eVm

Note that IBS has a strong effect since \( \epsilon_x/\epsilon_{x0} = 4.1, \epsilon_y/\epsilon_{y0} = 1.54 \) and \( \sigma_s \sigma_p/(\sigma_{x0} \sigma_{x0}) = 1.82 \). Nevertheless, the correction scheme of the damping ring presented in this chapter allows us to compensate efficiently alignment imperfections and to restore the beam emittances to the values achieved for the ideal machine (see Table 4.9).

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7.6 Dynamic aperture after correction

We have studied the dynamic aperture limitation for on-momentum particles due to alignment errors taking into account wiggler nonlinearities, although, they have negligible influence on the dynamic aperture (as it was investigated in Chapter 6).

Distortion of the dynamic aperture of the damping ring after dipole correction carried out in the presence of quadrupole misalignments only, with \(\langle \Delta Y_{\text{quad}} \rangle = \langle \Delta X_{\text{quad}} \rangle = 90 \, \mu\text{m} \) is shown in Fig. 7.13a. The curves of dynamic aperture plotted by grey color result from the corrections which were computed for 8 different samples of error distributions along the ring. The thick solid line represents the dynamic aperture without any alignment errors.

Distortion of the dynamic aperture after skew quadrupole and dipole correction carried out in the presence of all alignment errors listed in Table 7.1 is shown in Fig. 7.13b. The simulations were done for 8 different samples of error distributions along the ring. The mean value of the dynamic aperture indicated by the red line was obtained after the correction of the COD, CTCOs, residual vertical dispersion, and betatron coupling, – carried out by 246 horizontal, 146 vertical dipolar correctors and 48 skew quadrupole correctors.

As one can see by comparing Fig. 7.13a and Fig. 7.13b, the limitation of the dynamic aperture after correction is mainly determined by sextupole misalignments.

![Figure 7.13: Dynamic aperture of the damping ring after dipole correction carried out in the presence of quadrupole misalignments only, with \(\langle \Delta Y_{\text{quad}} \rangle = \langle \Delta X_{\text{quad}} \rangle = 90 \, \mu\text{m} \). The thick solid line shows the dynamic aperture without errors; b) Dynamic aperture of the damping ring after dipole and skew quadrupole correction carried out in the presence of all alignment errors listed in Table 7.1. The red thin line corresponds to the mean value of the dynamic aperture.](image)

Figure 7.13: a) Dynamic aperture of the damping ring after dipole correction carried out in the presence of quadrupole misalignments only, with \(\langle \Delta Y_{\text{quad}} \rangle = \langle \Delta X_{\text{quad}} \rangle = 90 \, \mu\text{m} \). The thick solid line shows the dynamic aperture without errors; b) Dynamic aperture of the damping ring after dipole and skew quadrupole correction carried out in the presence of all alignment errors listed in Table 7.1. The red thin line corresponds to the mean value of the dynamic aperture.
Chapter 8
Collective effects in the CLIC damping rings

The small emittance, short bunch length, and high current in the CLIC damping ring could give rise to collective effects which degrade the quality of the extracted beam. In this chapter, we survey a number of possible instabilities and estimate their impact on the ring performance. The effects considered include fast beam-ion instability, coherent synchrotron radiation, Touschek scattering, intrabeam scattering, resistive-wall wake fields, and electron cloud.

The design parameters of the CLIC damping ring are summarized in Tables 4.8 and 4.9. The limitations encountered at storage rings with similar features are manifold, ranging from microwave instability (SLC DR), over ion effects (SLC DR, ATF, KEKB, PEP-II), electron cloud (KEKB, PEP-II, BEPC, CESR, DAFNE), intrabeam scattering (ATF), and transverse coupled-bunch instabilities (KEKB, DAFNE).

8.1 Longitudinal and transverse μ-wave instability

For $b > \sigma_z$, the Keil-Schnell-Boussard threshold is [123]:

$$\frac{Z_{||}}{n} = Z_0 \sqrt{\frac{\pi}{2}} \frac{\gamma \alpha_p \sigma_z^2 \sigma_s}{N_{bp} r_0} \left( \frac{b}{\sigma_s} \right)^2 = 2.87 \ \Omega,$$

but it would be only 65 mΩ without the suppression factor $(b/\sigma_s)^2$, where the beam-pipe radius $b \approx 11$ mm represents a weighted average for arcs and wiggles, $Z_0 \approx 377 \ \Omega$ the free-space impedance, $\gamma$ the relativistic factor, and $r_0 = 2.82 \times 10^{-15}$ m the classical electron radius. For comparison the KEKB LER ring has a design longitudinal impedance of $Z_{||}/n \approx 15$ mΩ, while a much larger impedance of $Z_{||}/n \approx 72$ mΩ was measured [124]. Linear scaling would give 196 mΩ or 943 mΩ, respectively, at the CLIC revolution frequency. This number is well below the above threshold estimate.

There is also a transverse coasting-beam instability associated with the transverse impedance. Again applying the Keil-Schnell-Boussard criterion, the threshold for this instability may be written:

$$Z_\perp = Z_0 \frac{\gamma \alpha_p \sigma_z \sigma_s \nu_y \omega_0}{N_{bp} r_0} \frac{1}{C} = 19.4 \ \text{MΩ/m},$$

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where $\omega_0 = 2\pi/T_0$ is the revolution frequency. Although the relationship is strictly true only for the resistive wall impedance, the transverse broad-band impedance is often assumed to be related to the longitudinal broad-band impedance through:

$$Z_\perp = \frac{2c}{\omega_0 b^2} \frac{Z_{\parallel}}{n}.$$  \hspace{1cm} (8.3)

### 8.2 Coherent synchrotron radiation

Coherent synchrotron radiation (CSR) can cause emittance blow up and microwave instability [125]. Typically the beam is unstable only in an intermediate frequency range, namely above the beam-pipe shielding cut off and below the threshold frequency for Landau damping, if such a regime exists.

A novel code was recently developed [127] to calculate CSR effects in a storage ring over many turns. The shielding is computed from the actual vacuum chamber boundaries (no ‘parallel-plate approximation’). At the moment only longitudinal CSR effects are included and CSR is treated only for the arc dipoles, not for the wigglers. However, it has been argued that the wiggler contribution is small [125]. The calculation uses a paraxial approximation, and the bunch shape is assumed not to change during the passage through a bending magnet (it does change from turn to turn under the influence of the CSR). Transient CSR components are automatically included and they are important for CLIC.

The initially Gaussian bunch is deformed under the influence of the CSR wake, shown in Fig. 8.1. The rms bunch length increases with increasing bunch charge. Figure 8.2 illustrates that for a beam-pipe radius of 2 cm the CSR microwave instability threshold is reached at about twice the nominal charge. Above the threshold the energy spread is no longer constant. For a beam-pipe radius of 4 cm the threshold would be only 20% above the nominal charge. Further results can be found in [128].

### 8.3 Space charge

Space-charge forces lead to a significant vertical tune shift, because of the large circumference and small vertical beam size. The incoherent space charge tune shift is

$$\Delta \nu_\gamma^{sc} = \frac{N_{bp} r_0}{(2\pi)^{3/2} \gamma^3 \sigma_s} \int_0^C \frac{\beta_y}{\sigma_y (\sigma_x + \sigma_y)} ds \approx 0.1,$$  \hspace{1cm} (8.4)

which is close to the maximum acceptable value [129]. It could be reduced by raising the beam energy.
Figure 8.1: Initial (dashed) and equilibrium CSR wake (solid) of the CLIC damping ring for an arc beam pipe radius of 2 cm and a bunch population of $3 \times 10^9$.

Figure 8.2: Rms bunch length (solid) and energy spread (dashed) as a function of bunch charge for an arc beam-pipe radius of 2 cm.
8.4 Ion instabilities

In order to assess the importance of ion effects, we employ analytical formulae. Singly-charged ions are trapped within a bunch train if their mass, in units of proton masses, exceeds a critical value \[ A_{\text{crit}} = \frac{N_{\text{bp}}L_{\text{sep}}r_p}{2\sigma_y(\sigma_x + \sigma_y)} \], \[(8.5)\]

where \( r_p \) the classical proton radius, \( L_{\text{sep}} \) the bunch spacing (for CLIC damping ring \( L_{\text{sep}} = 16 \text{ cm} \)), and \( \sigma_{x,y} \) the horizontal or vertical rms beam size. The ion-induced incoherent tune shift at the end of the train is

\[
\Delta Q_{\text{ion}} \approx N_{\text{bp}}k_{\text{bt}}r_0C\pi\sqrt{(\gamma\epsilon_x)(\gamma\epsilon_y)}\left(\frac{\sigma_{\text{ion}}p}{k_BT}\right),
\]

\[(8.6)\]

where \( k_{\text{bt}} \) designates the number of bunches per train, \( C \) the ring circumference, \( \epsilon_{x,y} \) the rms geometric emittances, \( \sigma_{\text{ion}} \) the ionization cross section, \( p \) the vacuum pressure, \( k_B \) Boltzmann’s constant, and \( T \) the temperature in Kelvin. In (8.6), the ion distribution after filamentation has been approximated by a Gaussian with transverse rms sizes equal to the rms beam sizes divided by \( \sqrt{2} \). However, the real ion distribution is not Gaussian, but rather resembles a “Christmas tree”, described by a \( K_0 \) Bessel function [131]. The maximum tune shift at the center of the bunch will therefore be larger than our estimate. Under the same approximation, the central ion density at the end of the bunch train is

\[
\rho_{\text{ion}} \approx \frac{N_{\text{bp}}k_{\text{bt}}\sigma_{\text{ion}}p}{\pi\sigma_x\sigma_yk_BT},
\]

\[(8.7)\]

Lastly, the exponential vertical instability rise time of the fast beam-ion instability is estimated as [132]

\[
\tau_{\text{FBII}} \approx \frac{\gamma\sigma_y\sigma_x}{N_{\text{bp}}k_{\text{bt}}r_0\beta_y\sigma_{\text{ion}}}\left(\frac{k_BT}{p}\right)\sqrt{\frac{8}{\pi}}\left(\frac{\sigma_{\epsilon}}{f_i}\right),
\]

\[(8.8)\]

where the spread of the vertical ion oscillation frequency \( f_i \) as a function of longitudinal position, \( \sigma_{\epsilon_i} \), has been taken into account, as well as the variation of the vertical ion oscillation frequency with horizontal position and the nonlinear component of the beam-ion force.

For the CLIC damping ring we assume a total pressure of 1 nTorr \((1.3 \times 10^{-7} \text{ Pa})\). This pressure is roughly consistent with the best values achieved at the KEK/ATF and with typical pressures at the KEKB HER. Both growth rate and tune shift linearly scale with the pressure. We also assume that 20% of this vacuum pressure is due to carbon monoxide (CO), the rest being dominated by hydrogen. The pressure is taken to be the same in the arcs, wigglers and straight sections of the damping ring, respectively.

The resulting analytical estimates by Eqs. (8.5–8.8) are compiled in Table 8.1 [133], invoking an ionization cross section for CO molecules of 2 Mbarn, and a 30% relative ion-frequency spread \( \sigma_{\epsilon_i}/f_i \). Also, when estimating the instability rise time and ion-induced tune shift, we have, for simplicity, assumed trapping of CO ions along the train for all regions of the rings.

We have only considered the ions produced during the passage of a single train. To avoid ion accumulation between trains, the inter-train gap must be larger than

\[
L_{g,\text{cl}} \approx 10 \times c/(\pi f_i),
\]

\[(8.9)\]

with \( c \) the velocity of light. Values for the minimum clearing gap between trains, \( L_{g,\text{cl}} \), are also listed in Table 8.1. For the CLIC damping ring, clearing gaps of a few meters are sufficient. For
operating regime with 14 stored bunch trains, a gap between trains in the CLIC damping ring is not less than 7.5 m.

Table 8.1: Estimates for the incoherent tune shift and exponential fast beam-ion instability rise time for the CLIC damping ring. A partial CO pressure of 0.2 ntorr is assumed.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>CLIC Arc</th>
<th>Wiggler</th>
</tr>
</thead>
<tbody>
<tr>
<td>Critical mass, $A_{\text{crit}}$</td>
<td>15</td>
<td>9</td>
</tr>
<tr>
<td>Vertical ion frequency [MHz]</td>
<td>360</td>
<td>275</td>
</tr>
<tr>
<td>Minimum gap, $L_{g,\text{cl}}$ [m]</td>
<td>2.7</td>
<td>3.5</td>
</tr>
<tr>
<td>Ion density $\rho_{\text{ion}}$ [cm$^{-3}$]</td>
<td>0.58</td>
<td>0.34</td>
</tr>
<tr>
<td>Exponential rise time</td>
<td></td>
<td></td>
</tr>
<tr>
<td>at train end [μs]</td>
<td>189</td>
<td>185</td>
</tr>
<tr>
<td>[av. rise t. 187]</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Incoherent tune shift</td>
<td>0.001</td>
<td>0.001</td>
</tr>
<tr>
<td>at train end $\Delta Q_y$</td>
<td></td>
<td>[total 0.0026]</td>
</tr>
</tbody>
</table>

Figure 8.3: Simulated vertical trajectories for CO ions during the passage of 17.6-m long CLIC bunch trains separated by 7.5 m (left) and for H ions and half of the first train (right).

Complementary to the above analytical estimates, the ion trapping condition, the survival between trains, and the evolution of the central ion density in simulations using a newly developed computer code [134] have been explored.

The simulations were performed for an arc section of the CLIC damping ring considering a partial pressure of 0.1 ntorr and 2 Mbarn ionization cross section. Figure 8.3 shows sample trajectories in the $x-z$ plane for CO (left) and H ions (right). The hydrogen ions are overfocused between bunches of the train, and most of them are quickly lost to the wall, while the CO ions perform stable oscillations, which is consistent with Eq. (8.5). Figure 8.4 (left) shows the central CO-ion density evolution. The final density value at the end of the train is about 2.5 times higher.
than predicted by our analytical formula, which we attribute to the non-Gaussian shape of the real ion distribution. Some of the hydrogen ions re-stabilize at large amplitudes, under the influence of the nonlinear beam field, and they are not lost to the chamber wall during the train passage, as indicated in Fig. 8.4 (right).

The simulation confirms that in CLIC only a small fraction of CO ions survive from train to train for inter-train gaps larger than 3 m, consistent with our estimate. For operating regime with 4 stored bunch trains, a gap between trains in the CLIC damping ring is 73.8 m, i.e., more than 20 times the minimum gap needed for ion clearing. In this case, the residual ion population from the previous train is negligible.

![Image](image.png)

Figure 8.4: Simulated evolution of central ion density along a CLIC bunch train (left); transverse H ion distribution during single-train passage (right).

### 8.5 Electron cloud

Electron-cloud effects in the CLIC positron damping ring were discussed by Frank Zimmermann in Refs [135, 136]. In the arcs, antechambers absorb the entire photon flux. In the wiggler section, a residual photon flux of about $3 \times 10^{18} \text{ m}^{-1}\text{s}^{-1}$ or about 3 photons per passing positron per meter length (about 30% of the emitted ones) do not enter the antechamber. The average photon energy is about 2.2 keV. Simulated electron densities in the wiggler vary between $10^{13} \text{ m}^{-3}$ and several $10^{14} \text{ m}^{-3}$, which is to be compared with a simulated single-bunch instability threshold of about $2 \times 10^{12} \text{ m}^{-3}$. This implies that special measures must be taken to reduce the electron density, such as the installation of dedicated photon stops intercepting the straight-ahead radiation, and the application of electric clearing fields.

### 8.6 Touschek lifetime

The ultra-low transverse emittances are achieved with an RF voltage close to the energy loss per turn. This implies a small momentum acceptance, so that the lifetime of the stored beam is limited by the Touschek effect. The Touschek lifetime can be used as a diagnostics for emittance tuning and
The Touschek lifetime can be computed using the Piwinski formalism of [139], including horizontal and vertical dispersion, which was implemented in the MAD-X programme [140]. Figure 8.5 illustrates how the Touschek lifetime varies with the ring RF voltage [141], even for an RF voltage as low as 2.5 MV, the Touschek lifetime is longer than the bunch-train store time of 46.6 ms corresponding to the operation regime with 14 bunch trains (store time of 13.3 ms corresponding to the operation regime with 4 bunch trains). A slight increase in rf voltage raises the beam lifetime substantially, which can be exploited for ring-tuning purposes.

![Figure 8.5: Touschek lifetime as a function of RF voltage for a bunch population of $3.1 \times 10^9$.](image)

### 8.7 Resistive wall

The dominant transverse impedance source is the resistive wall in the long wiggler sections with only about 8 mm vertical half aperture. The classical growth rate of the most unstable mode is estimated as

$$ \frac{1}{\tau_{rw}} \approx \frac{1}{2} \frac{\pi^2}{8} \frac{\beta_y N_{bp} h r_0 e^2}{2\pi b_w^3 \gamma \sqrt{\sigma C}} \frac{1}{\sqrt{|Q - n|}} \approx 1854 \text{s}^{-1}, $$

(8.10)

where we have introduced the factor $\pi^2/8$ to account for the flat chamber and another factor 1/2, since the wigglers occupy about half the circumference. The parameter $h = 2281$ is the harmonic number. The ring was pessimistically assumed to be completely filled with $h$ equidistant bunches. Also, we have taken the resistivity of copper $\sigma \approx 5.4 \times 10^{17} \text{s}^{-1}$ and a fractional tune below the half integer, choosing $|Q - n| \approx 0.85$ for the most unstable coupled bunch mode. The classical resistive-wall growth time of 590 $\mu$s corresponds to about 500 turns.

### 8.8 Coupled-bunch instabilities

Higher-order modes (HOMs) in the RF cavities could drive narrow-band transverse or longitudinal instabilities, as have been observed in many storage rings. These may be avoided by a careful design and dedicated HOM dampers. The average beam current in the CLIC damping ring is much lower than that reached at the two B factories.
Chapter 9
Summary

• Three variants (RING-1, RING-2, RING-3) of the linear optics for the CLIC damping ring design have been considered. The general lattice parameters of these designs are listed in Table 4.8 while the parameters of the extracted beam are listed in Table 4.9. In all three designs, the damping ring is composed of two long dispersion free FODO-cell straight sections with wigglers, two TME-cell arcs, and four dispersion suppressors connecting the arcs and the straights, forming a racetrack shape. There are only two differences between these designs which are 1) the number of the wiggler FODO cells and 2) the wiggler parameters. Other block-structures such as the arc, wiggler FODO cell, dispersion suppressor, beta-matching section, and injection/extraction region are the same, as described in Sections (4.2), (4.4), (4.5), and (4.6.1).

• The RING 1 design is optimized for the NdFeB permanent magnet wiggler with $\lambda_w = 10 \text{ cm}$ and $B_w = 1.7 \text{ T}$. The straight sections comprise 76 NdFeB wiggler magnets. The ring circumference is equal to 364.96 m. The RING 2 design is similar to the RING 1, but superconducting Nb$_3$Sn wigglers ($\lambda_w = 4.5 \text{ cm}$ and $B_w = 2.52 \text{ T}$) are used instead of the NdFeB wigglers. In the RING 3 the same superconducting Nb$_3$Sn wigglers are used but their number is reduced to 48 units, which shortens the circumference of the ring to 300.48 m.

• Taking into account the effect of IBS, the RING-2 and RING-3 designs meet the principal specifications for extracted beam emittance and damping time which are listed in Table 4.2. The RING-1 with the NdFeB permanent wigglers produces the transverse emittances $\gamma \epsilon_x = 540 \text{ nm}$ and $\gamma \epsilon_y = 3.4 \text{ nm}$ which are larger than the target values by 20% and 13%, respectively.

• In spite of the fact that the transverse emittances in the RING 1 design are larger than the transverse emittances in the RING 2 and RING 3 designs, the damping ring design RING 1 with the NdFeB permanent magnet wigglers was studied in detail because a concrete design for the NdFeB permanent wiggler with $\lambda_w = 10 \text{ cm}$ and $B_w = 1.7 \text{ T}$ was developed while writing this thesis. In particular, the field map for this wiggler was known, which allowed detailed studies of the nonlinear wiggler effect on the dynamic aperture and of the sensitivity of the machine to alignment errors. A tentative design of the superconducting Nb$_3$Sn wigglers was suggested only recently. For this reason, the superconducting wiggler scenarios were not studied in the framework of the present thesis.
• The CLIC damping ring features a lattice with very strong focusing to meet requirements for the ultra-low target beam emittance. The average value of betatron and dispersion functions in the arc are small \((\langle \beta_x \rangle = 0.85 \, \text{m}, \langle \beta_y \rangle = 2.2 \, \text{m} \text{ and } \langle D_x \rangle = 0.0085 \, \text{m} \)). As a consequence, to compensate the large natural chromaticity with the small optical functions, the strength of the sextupoles located in the arcs becomes very strong, which limits the dynamic aperture of the machine. A non-interleaved \(-I\) arrangement of the sextupole pairs cannot be applied because of their intolerable strength. Nine interleaved sextupole families arranged so as to form a second order sextupolar achromat were used for the chromaticity correction and at the same time for maximizing the dynamic aperture. A dynamic aperture of \(7\sigma_x^{\text{inj}}\) horizontally and \(14\sigma_y^{\text{inj}}\) vertically in terms of injected beam size was obtained for the damping ring lattice.

• The nonlinearities introduced by the NdFeB wigglers do not lead to a reduction of the dynamic aperture when the sextupoles are turned on. These nonlinearities are negligible in comparison with the nonlinearities produced by the sextupoles.

• In order to limit the synchrotron radiation hitting the vacuum chamber in the straight wiggler sections, an effective collimation system was developed. A copper absorber cooled by water is located after each wiggler. Such configuration of regularly distributed absorbers ensures the absorption of 334.5 kW of SR power per straight section, for an average current of 0.52 A corresponding to the maximum number of bunch trains which can be stored in the damping ring. The rest of the SR power, 90.3 kW, will be taken up by a terminal absorber placed at the end of the straight section. Only a small fraction of SR power hits the vacuum chamber. Its integrated value over the vacuum chamber of the straight section is equal to 6 W/m for the closed orbit distortion of 100 \(\mu\text{m}\).

• Without any correction, already fairly small vertical misalignments of the quadrupoles and, in particular, the sextupoles, introduce unacceptable distortions of the closed orbit as well as intolerable spurious vertical dispersion and betatron coupling. An effective correction scheme was developed. The correction of the closed orbit distortion (COD), cross-talk between vertical and horizontal closed orbits (CTCOs), residual vertical dispersion and betatron coupling is carried out by 246 horizontal and 146 vertical dipolar correctors as well as 48 skew quadrupole correctors. For the alignment errors listed in Table 7.1, the correction system restores the transverse emittances to the same values \(\gamma\epsilon_y = 3.4 \, \text{nm} \text{ and } \gamma\epsilon_x = 540 \, \text{nm} \text{ (taking into account IBS) as achieved for the ideal machine (without any imperfections). A dynamic aperture of } 5\sigma_x^{\text{inj}} \text{ horizontally and } 9\sigma_y^{\text{inj}} \text{ vertically in terms of injected beam size is obtained after the correction.}

• The CLIC damping ring operates well below the longitudinal microwave and transverse mode-coupling thresholds. Coherent synchrotron radiation is benign, causing only a 5% bunch lengthening without instability. Intrabeam scattering was incorporated in the design optimization and the target emittances are reached including its effect. The Touschek lifetime is acceptable and can easily be increased for beam-tuning purposes, if desired. The resistive-wall instability driven by the impedance of the wiggler chamber can be suppressed by a feedback system. The space-charge tune shift is close to the limit considered acceptable. Potential limitations to be addressed are the high electron-cloud densities in the wiggler sections and the fast beam-ion instability. Possible remedies include clearing electrodes and photon stops for the wiggler, and an improved vacuum.
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Appendix A

Transformation matrices for accelerator magnets

The coefficients $C(s)$, $S(s)$, $C'(s)$, $S'(s)$, $\hat{D}(s)$ and $\hat{D}'(s)$ of transformation matrix (2.7) can be expressed not only in terms of Twiss parameters as it was done in Eq. (2.11) but also in terms of magnetic field properties such as strength of the dipole field, gradient of quadrupole field, length or bending angle produced by magnetic-optics elements. The solution for the complete lattice or for the desired sequence of optical elements is just the consecutive product of their individual matrices. We assume that field of a magnet is independent of $s$ inside the magnet and drops abruptly to zero at the ends of magnet (hard-edge model).

The bending sector magnet has magnet end faces which are perpendicular to the circular trajectory of particles. The magnetic field of this magnet is explicitly defined by three parameters: $K_1$, $\theta = L/\rho$ and $\rho$. Assuming $K_1 = 0$, the transfer matrix of the sector magnet is the following

$$
M_x = \begin{pmatrix} C_x & S_x & \hat{D}_x \\ C'_x & S'_x & \hat{D}'_x \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \cos \theta & \rho \sin \theta & \rho(1 - \cos \theta) \\ -\frac{1}{\rho} \sin \theta & \cos \theta & \sin \theta \\ 0 & 0 & 1 \end{pmatrix}
$$

$$
M_y = \begin{pmatrix} C_y & S_y & \hat{D}_y \\ C'_y & S'_y & \hat{D}'_y \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & l & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}
$$

where $\hat{D}_x$ and $\hat{D}'_x$ are defined by Eq. (2.15). As one can see, a dipole sector magnet does not disturb the vertical motion. The Twiss parameters $\beta$, $\alpha$ and $\gamma$ at the exit of the magnets are found by Eq. (2.13) and the dispersion as

$$
\begin{pmatrix} D \\ D' \\ 1 \end{pmatrix} = M_x \begin{pmatrix} D_0 \\ D'_0 \\ 1 \end{pmatrix}
$$

where the index "0" refers to the entrance of the bending magnet. For the small bending angle
\[ \theta \ll 1 \text{ the transfer matrix } M_x \text{ can be approximated as} \]

\[
M_x = \begin{pmatrix}
\cos \theta & \rho \sin \theta & \rho (1 - \cos \theta) \\
\frac{-1}{\rho} \sin \theta & \cos \theta & \sin \theta \\
0 & 0 & 1
\end{pmatrix} \approx \begin{pmatrix}
1 & L & \rho (1 - \cos \theta) \\
0 & 1 & \sin \theta \\
0 & 0 & 1
\end{pmatrix}
\]

Using Eqs. (2.11) and (A.1–A.2), the transformation of the horizontal lattice functions through a non-focusing \((K_1 = 0)\) sector bending magnet with length \(L\) and small bending angle \(\theta \ll 1\) is given by

\[
\begin{align*}
\beta(s) &= \beta_0 - 2\alpha_0 s + \gamma_0 s^2 \\
\alpha(s) &= \alpha_0 - \gamma_0 s \\
\gamma(s) &= \gamma_0 \\
D(s) &= D_0 + D'_0 s + \rho_0 (1 - \cos \theta) \\
D'(s) &= D'_0 + \sin \theta
\end{align*}
\]

where the index "0" refers to the entrance of the bending magnet.

The transfer matrices of other important magnets are given below

- **Drift space** \(\frac{1}{\rho} = 0, K_1 = 0\), length - \(L\)

\[
M_x = M_y = \begin{pmatrix}
1 & L & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\]

- **Quadrupole** \(\frac{1}{\rho} = 0, K_1 > 0\), length - \(L\), \(\varphi = L\sqrt{|K_1|}\)

\[
M_x = \begin{pmatrix}
\cos \varphi & \frac{1}{\sqrt{|K_1|}} \sin \varphi & 0 \\
-\sqrt{|K_1|} \sin \varphi & \cos \varphi & 0 \\
0 & 0 & 1
\end{pmatrix}, \quad M_y = \begin{pmatrix}
\cosh \varphi & \frac{1}{\sqrt{|K_1|}} \sinh \varphi & 0 \\
\sqrt{|K_1|} \sinh \varphi & \cosh \varphi & 0 \\
0 & 0 & 1
\end{pmatrix}
\]

These matrices describe horizontal focusing and vertical defocusing. For \(K_1 < 0\), the matrices \(M_x\) and \(M_y\) are interchanged and we get horizontal defocusing and vertical focusing.

- **Dipole rectangular magnets** are often built straight with the magnet end plates not perpendicular to the central trajectory which introduces slight focusing in the vertical planes. For \(\theta = L/\rho, \delta = \varphi/2\)
\[ M_x = \begin{pmatrix} 1 & \rho \sin \theta & \rho(1 - \cos \theta) \\ 0 & 1 & 2 \tan \frac{\theta}{2} \\ 0 & 0 & 1 \end{pmatrix}, \quad M_y = \begin{pmatrix} \cos \theta & \rho \sin \theta & 0 \\ -\frac{1}{\rho} \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \]

(A.6)
Appendix B

Second order chromaticity

The chromatic terms of second order, which are independent of the angle variables, drive the second order chromaticity. Effective quadrupole and sextupole strengths experienced by an off-momentum particle can be expressed as:

\[ K_1(s, \delta) = \frac{K_1(s)}{1 + \delta}, \quad K_2(s, \delta) = \frac{K_2(s)}{1 + \delta} \]  

(B.1)

where \( K_1(s) \) and \( K_2(s) \) are the normalized (divided by magnetic rigidity \( B\rho \)) quadrupole and sextupole gradients for on-momentum particles. Horizontal dispersion, beta functions and the tune shift in terms of the first and second order chromaticity \( \xi^{(1)} \) and \( \xi^{(2)} \), respectively, can be expanded as power series in \( \delta \)

\[
\begin{align*}
D_x(s, \delta) &= D_x^{(0)}(s) + \Delta D_x^{(1)}(s)\delta + \Delta D_x^{(2)}(s)\delta^2 + O(\delta^3) \\
\beta(s, \delta) &= \beta^{(0)}(s) + \Delta \beta^{(1)}(s)\delta + \Delta \beta^{(2)}(s)\delta^2 + O(\delta^3) \\
\Delta \nu &= \xi^{(1)}(\delta) + \xi^{(2)}(\delta^2) + O(\delta^3)
\end{align*}
\]  

(B.2)

where the superscript \((1), (2)\ldots(n)\) for the dispersion and beta functions denotes a chromatic expansion. The second order chromaticity may be expressed by considering the parameter dependence in the formula for the linear chromaticity Eq. (5.25) with respect to \( \delta \) as:

\[
\begin{align*}
\xi_x^{(2)} &= \frac{1}{2} \left[ \frac{\partial \nu_x(\delta)}{\partial \delta} \right]_{\delta=0} = \frac{1}{2} \left[ \frac{\partial \nu_x(\delta)}{\partial \delta} \right]_{\delta=0} \\
&= -\frac{1}{8\pi} \int_0^C \left[ \frac{\partial K_1(s, \delta)}{\partial \delta} - \frac{\partial K_2(s, \delta)}{\partial \delta} D_x^{(0)}(s) \right] \beta_x^{(0)}(s) ds \\
&\quad + \frac{1}{8\pi} \int_0^C \left[ K_2(s) \frac{\partial D_x(s, \delta)}{\partial \delta} \beta_x^{(0)}(s) - \left[ K_1(s) - K_2(s) D_x^{(0)}(s) \right] \frac{\partial \beta_x(s, \delta)}{\partial \delta} \right] ds
\end{align*}
\]  

(B.3)

Substituting Eq. (B.1) and Eq. (B.2) into Eq. (B.3), we obtain the second order chromaticity \( \xi_x^{(2)} \) and \( \xi_y^{(2)} \),

\[
\begin{align*}
\xi_x^{(2)} &= -\frac{1}{2} \xi_x^{(1)} + \frac{1}{8\pi} \int_0^C \left[ K_2 \Delta D_x^{(1)} \beta_x^{(0)} - \left[ K_1 - K_2 D_x^{(0)} \right] \Delta \beta_x^{(1)} \right] ds \\
\xi_y^{(2)} &= -\frac{1}{2} \xi_y^{(1)} - \frac{1}{8\pi} \int_0^C \left[ K_2 \Delta D_x^{(1)} \beta_y^{(0)} + \left[ K_1 - K_2 D_x^{(0)} \right] \Delta \beta_y^{(1)} \right] ds
\end{align*}
\]  

(B.4)
where second order dispersion $\Delta D_x^{(1)}$ is defined by

$$
\Delta D_x^{(1)}(s) = \frac{\sqrt{\beta_x^{(0)}(s)}}{2 \sin (\pi \nu_x)} \int_s^{s+C} \sqrt{\beta_x^{(0)}(s')} \left[ K_1(s') - K_2(s') D_x^{(0)}(s') \right] \times D_x^{(0)}(s') \cos(|\mu_x(s') - \mu_x(s)| - \pi \nu_x) ds'
$$

(B.5)

and the beta-beat functions $\Delta \beta_x^{(1)}$ and $\Delta \beta_y^{(1)}$ are defined by

$$
\Delta \beta_x^{(1)}(s) = \frac{\beta_x^{(0)}(s)}{2 \sin (2 \pi \nu_x)} \int_s^{s+C} \beta_x^{(0)}(s') \left[ K_1(s') - K_2(s') D_x^{(0)}(s') \right] \times \cos(2|\mu_x(s') - \mu_x(s)| - 2 \pi \nu_x) ds'
$$

(B.6)

$$
\Delta \beta_y^{(1)}(s) = -\frac{\beta_y^{(0)}(s)}{2 \sin (2 \pi \nu_y)} \int_s^{s+C} \beta_y^{(0)}(s') \left[ K_1(s') - K_2(s') D_x^{(0)}(s') \right] \times \cos(2|\mu_y(s') - \mu_y(s)| - 2 \pi \nu_y) ds'
$$

(B.7)

As one can see from the Eq. (B.6) and Eq. (B.7), if the phase advance is equal to $\pi/2$ between two sources of chromaticity with equal strength $(K_1 l_1 \beta(s_1) = K_1 l_2 \beta(s_2)$ for quadrupoles or $K_2 l_1 \beta(s_1) D_x(s_1) = K_2 l_2 \beta(s_2) D_x(s_2)$ for sextupoles), the resulting beta-beat $\Delta \beta^{(1)}$ will be zero, since $\cos(-2\pi \nu) + \cos(2|\pi/2| - 2 \pi \nu) = 0$. Alternatively, from Eq. (B.5), if we want to cancel the dispersion $\Delta D_x^{(1)}$, the two sources should be separated by phase advance of $\pi$. However, in this case the $\Delta \beta^{(1)}$ will add exactly in phase.

Hence to reduce the second order chromaticity, the first order changes in the beta functions and in the dispersion should be minimized. Conversely, the regions where $\Delta \beta^{(1)}$ and $\Delta D_x^{(1)}$ are large will contribute the most to the second order chromaticity. The above expressions also exhibit the variation of $\xi^{(2)}$ with the global betatron tune. Since the $\Delta \beta^{(1)}$ diverges at integer and half-integer resonances, $\xi^{(2)}_{x,y}$ will be amplified when the global betatron tune $\nu_{x,y}$ will be closed to integer or half-integer value and will be a minimum when $\nu_{x,y}$ will be equal to a quarter integer.