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4 An elementary property of CTs

1 Projective null cone

In the previous lecture we motivated the study of Scale Invariant (SI) fixed points. We also argued that the SI of the Renormalization Group (RG) fixed points is in general enhanced to Conformal Invariance (CI) = invariance under coordinate transformations that leave the metric invariant up to a coordinate-dependent scale factor $\lambda(x)$

$$x \rightarrow x', \text{ such that } dx'^2 = \lambda(x) dx^2 .$$

This means that the Jacobian of the coordinate transformation should have the special form

$$J = \frac{\partial x'^\mu}{\partial x^\nu} = b(x) O_\mu^\nu(x) ,$$

where $b(x)$ a scale factor and $O^\mu_\nu$ a rotation matrix, that both may depend on the coordinates. Then,

$$\lambda(x) = b(x)^2$$
We also saw that in $D \geq 3$ the group of Conformal Transformations (CT) is finite dimensional and is generated by the Poincaré transformations plus dilatations plus Special Conformal Transformations (SCT).

Then we introduced the concept of primary operators, which transform under CT as

$$\phi(x) \rightarrow \tilde{\phi}(x') = \frac{1}{b(x)^\Delta} \phi(x), \quad (1.4)$$

if $\phi$ is scalar, or

$$\phi(x) \rightarrow \tilde{\phi}(x') = \frac{1}{b(x)^\Delta} \mathcal{R}[O_{\nu}(x)] \phi(x), \quad (1.5)$$

if $\phi$ has intrinsic spin, i.e. belongs to an irreducible representation $\mathcal{R}$ of $SO(D)$. The correlation functions of $\tilde{\phi}$ are the same as those of $\phi$ ($\tilde{\phi}$ can be thought of as an image of $\phi$ under a non-uniform RG transformation). In the future we will sometimes omit the tilde from the start.

Operationally, the above transformation property simply means that the n-point correlation functions of $\phi$ must satisfy

$$\langle \phi(x')\phi(y') \ldots \rangle = \frac{1}{b(x)^\Delta} \frac{1}{b(y)^\Delta} \ldots \langle \phi(x)\phi(y) \ldots \rangle. \quad (1.6)$$

This condition is clearly an important constraint on the correlation functions of the theory, and in this lecture we will study its consequences. There are several ways to achieve that, some more pedestrian than others. We will choose a method which gives us the most information in the least possible time.

First of all we have to understand better the conformal algebra. Last time we wrote the formulas for the vector fields that correspond to the generators of the group

$$P_\mu = i\partial_\mu \rightarrow \text{translations},$$

$$M_{\mu\nu} = i(x_\mu \partial_\nu - x_\nu \partial_\mu) \rightarrow \text{rotations},$$

$$D = ix^\mu \partial_\mu \rightarrow \text{dilatations},$$

$$K_\mu = i(2x_\mu(x^\nu \partial_\nu) - x^2 \partial_\mu) \rightarrow \text{SCT}. \quad (1.7)$$

Since we know the precise expressions for the generators, we can now easily compute their commutators. We find some relations which correspond to the Poincare algebra

$$[M_{\mu\nu}, M_{\rho\sigma}] = -i(\delta_{\mu\rho}M_{\nu\sigma} \pm \text{permutations}),$$

$$[M_{\mu\nu}, P_\rho] = i(\delta_{\nu\rho}P_\mu - \delta_{\mu\rho}P_\nu), \quad (1.8)$$

The interesting new relations are

$$[D, P_\mu] = -iP_\mu,$$

$$[D, K_\mu] = iK_\mu,$$

$$[P_\mu, K_\nu] = 2i(\delta_{\mu\nu}D - M_{\mu\nu}). \quad (1.9)$$
It turns out that this conformal algebra is isomorphic to \( SO(D+1, 1) \), the algebra of Lorentz transformations in \( \mathbb{R}^{D+1, 1} \) Minkowski space.

Consider in the latter space the coordinates

\[ X^1, \ldots, X^D, X^{D+1}, X^{D+2} \quad (1.10) \]

where \( X^{D+2} \) is the timelike direction. We will also use the lightcone coordinates

\[ X^+ = X^{D+2} + X^{D+1}, \; X^- = X^{D+2} - X^{D+1}. \quad (1.11) \]

In terms of the above, the mostly plus metric \( \eta_{MN} \) in \( \mathbb{R}^{D+1, 1} \) is

\[ ds^2 = \sum_{i=1}^{D} (dX^i)^2 - dX^+ dX^- . \quad (1.12) \]

We identify the generators of the conformal algebra with the generators of \( SO(D + 1, 1) \) as follows

\[ J_{\mu\nu} = M_{\mu\nu}, \]
\[ J_{\mu+} = P_\mu, \]
\[ J_{\mu-} = K_\mu, \]
\[ J_{++} = D, \quad (1.13) \]

with \( \mu, \nu = 1, \ldots, D \). It is understood that \( J_{\mu\nu} \) is antisymmetric in the interchange of \( \mu \) and \( \nu \). Then one can check that the conformal algebra commutation relations coincide with those of the \( SO(D + 1, 1) \) algebra:

\[ [J_{MN}, J_{RS}] = -i(\eta_{MR}J_{NS} \pm \text{permutations}) \quad (1.14) \]

One example is

\[ [J_{\mu+}, J_{\nu-}] \propto \delta_{\mu\nu} J_{++} + \delta_{+-} J_{\mu\nu} . \quad (1.15) \]

**Exercise:** Check and fix all the constants in the above identification.

This result means that the conformal group, which acts in a non-trivial way on the \( D \) dim space, acts naturally (linearly) on the \( \mathbb{R}^{D+1,1} \) space. In the vector representation we can write

\[ X^M \rightarrow \Lambda^M_N X^N , \quad (1.16) \]

with \( \Lambda^M_N \) an \( SO(D+1,1) \) matrix. If we could somehow get an action on \( \mathbb{R}^D \) out of this simple action, then the implications of CI would be easier to understand. To do that however, we have to embed the \( D \) dimensional space into the \( D + 2 \) dimensional space, that is to get rid of the two extra coordinates.

To get rid of one of the coordinates let’s restrict the attention to the null cone:

\[ X^2 = 0 \quad (1.17) \]

in the \( D + 2 \) dimensional spacetime. Since this constraint is preserved by the action of the group, we don’t lose simplicity.
To get down to $D$ dimensions, we take a generic section of the light-cone:

$$X^+ = f(X^\mu), \quad (1.18)$$

The section is parametrized by $X^\mu$ which we identify with the $\mathbb{R}^D$ coordinates $x^\mu$

The group $SO(D + 1, 1)$ acts on the section as follows (see Fig. 1). A point $x^\mu$ on the section and defines a lightray. If we apply a Lorentz transformation, this lightray will be mapped into a new one which passes through another point $x'^\mu$. Thus

$$x^\mu \rightarrow \text{light ray} \xrightarrow{\Lambda^\mu{}_{\nu} \in SO(D+1,1)} \text{light ray'} \rightarrow x'^\mu, \ x \in \mathbb{R}^D. \quad (1.19)$$

Figure 1: Red: section, blue: light ray and light ray'.

Figure 2: How the infinitesimal interval transforms under the defined $SO(D + 1, 1)$ action.

While this defines an action of $SO(D + 1, 1)$ on $\mathbb{R}^D$, we have to check if this action corresponds to a CT. Consider the metric $ds^2$ on the section induced from the Minkowski
metric in $\mathbb{R}^{D+1,1}$. We have:

$$ds^2 = dx^2 - dX^+ dX^- |_{X^+=f(x), X^-=-x^2/x^+} = g_{\mu\nu}(x)dx^{\mu} dx^{\nu}, \quad (1.20)$$

where $g_{\mu\nu}(x)$ is a metric we could compute explicitly in terms of $f(x)$ but we won’t need it.

The action of $SO(D+1,1)$ on a point $x$ can be split into two steps, 1) $X \rightarrow \Lambda X$ and 2) then rescale to get back into the section. We want to understand how this action changes the infinitesimal interval length (see Fig.2). The first step is an isometry and does not change $ds^2$. The second step changes the metric by an $x$-dependent scale factor. Indeed, assuming that we have to rescale by $\lambda$ to get back into the section, where $\lambda$ in general depends on $X$, we have:

$$(d(\lambda(X)X))^2 = (\lambda dX + X(\nabla \lambda dX))^2 = \lambda^2 dX^2 \quad (1.21)$$

the other terms vanishing by $X^2 = 0, X.dX = 0$.

We conclude that the metric transformation is of the form:

$$ds'^2 = c(x) ds^2, \quad c(x) = \lambda(X)^2. \quad (1.22)$$

This will exactly agree with the definition of the conformal transformations as long as $ds^2$ is flat. From the definition of $ds^2$ it’s easy to guess which $f(X)$ achieves this: it is $f(X) = \text{const}$, so that $dX^+ = 0$. For simplicity and without loss of generality we take this constant equal to 1. Thus our Euclidean section is parametrized as:

$$X = (X^+, X^-, X^\mu) = (1, x^2, x^\mu) \quad (1.23)$$

We note in passing that by taking this section and rescaling it in the radial direction by an $x$-dependent factor we can reproduce any metric which is a Weyl transformation of the flat space metric (for example the metric on the sphere, de Sitter or Anti de Sitter spaces).

We now would like to extend the above action to fields. We thus consider fields $\phi(X)$ defined on the cone. The most natural action of the Lorentz group on such scalar fields is

$$X \rightarrow X', \quad \phi(X) \rightarrow \tilde{\phi}(X') = \phi(X). \quad (1.24)$$

The field on the euclidean section will be assumed to coincide with the $D$-dimensional field:

$$\phi(X)|_{\text{section}} = \phi(x) \quad (1.25)$$

Finally, we will assume that $\phi$ depends homogeneously on $X$:

$$\phi(\lambda X) = \lambda^{-\Delta} \phi(X), \quad (1.26)$$

Let us show that these conditions imply the correct transformation rule for the fields on $\mathbb{R}^D$

$$\phi(x') = b(x)^{-\Delta} \phi(x). \quad (1.27)$$

Indeed, $b(x)$ in this equation is the local expansion factor, and according to Eq. (1.22) it must be identified with $\lambda(X)$, the scale factor in the second phase of $SO(D + 1,1)$ action. Since $\phi(X)$ scales homogeneously with $\lambda$, we get exactly what we need.

Using this “projective light cone” formalism that we developed, any quantity (e.g. correlation function) which is conformally invariant in the $\mathbb{R}^D$ space, can be lifted to a quantity invariant under $SO(D + 1,1)$ in the $D + 2$ dimensional space. Basically, this formalism makes CI as simple as Lorentz invariance.
2 Simple applications

2.1 Primary scalar 2-point function

The expression of the two-point function on the light-cone is

\[ \langle \phi(X)\phi(Y) \rangle = \frac{c}{(X \cdot Y)^\Delta} , \tag{2.1} \]

with \( c \) a constant and \( \Delta \) the field’s scaling dimension. The above is the most general Lorentz invariant expression consistent with scaling of both \( \phi(X) \) and \( \phi(Y) \) with degree \( \Delta \). Note that \( X^2 = Y^2 = 0 \) cannot appear. To write the two-point function in the physical space, we project \( X \) and \( Y \) on the section, i.e.

\[ X = (X^+, X^-, X^\mu) = (1, x^2, x^\mu) \quad \text{and} \quad Y = (Y^+, Y^-, Y^\mu) = (1, y^2, y^\mu) . \tag{2.2} \]

We get

\[ X \cdot Y = X^\mu Y_\mu - \frac{1}{2} (X^+ X^- + X^- Y^+) \]
\[ = x^\mu y_\mu - \frac{1}{2} (x^2 + y^2) \]
\[ = -\frac{1}{2} (x - y)^2 . \tag{2.3} \]

The two-point function (2.1) is therefore projected to

\[ \langle \phi(x)\phi(y) \rangle \propto \frac{1}{(x - y)^{2\Delta}} \tag{2.4} \]

That this expression is consistent with the SI of the field \( \phi(x) \) is obvious; we have shown that it is also CI. If we wanted to show it in a pedestrian way, without using the projective light cone, we would have to show that

\[ |x' - y'|^2 = b(x)b(y)|x - y|^2 . \tag{2.5} \]

for any CT, which does not look simple. The standard way to proceed is to show first that this holds for the inversion transformation

\[ x^\mu \rightarrow \frac{x^\mu}{x^2} , \tag{2.6} \]

which is a CT. Indeed the Jacobian is given by

\[ \frac{\partial x'^\mu}{\partial x^\nu} = \frac{1}{x^2} \left( \delta^{\mu\nu} - \frac{2x^\mu x^\nu}{x^2} \right) \equiv b_{inv}(x) I^{\mu\nu}(x) , \tag{2.7} \]

where \( b_{inv}(x) = 1/x^2 \) and \( I^{\mu\nu}(x) \) an orthogonal matrix. This can be easily seen if we go to a particular frame where \( x \) lies on the \( x_1 \) direction. Then the matrix is diagonal

\[ I^{\mu\nu}(x) = \begin{pmatrix} -1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 1 & 0 \end{pmatrix} , \tag{2.8} \]
and is clearly an $O(D)$ matrix. But it is not in $SO(D)$. This means that inversion is not in the connected part of the conformal group, i.e. it cannot be obtained by exponentiating a Lie algebra element.

If we apply inversion twice we get back to the connected component. In fact we can reproduce SCT this way:

$$SCT_\alpha = \text{inversion} \times \text{translation}_\alpha \times \text{inversion} . \quad (2.9)$$

Eq. (2.5) is not difficult to verify for inversion:

$$|x' - y'|^2 = \left| \frac{x^\mu - y^\mu}{x^2} \right|^2 = \frac{|x - y|^2}{x^2 y^2} = b_{\text{inv}}(x)b_{\text{inv}}(y)|x - y|^2 . \quad (2.10)$$

And then it holds for SCT and for all other CT’s by extension. This way of proof however looks a bit ad hoc, and becomes more and more awkward as we go to fields with spin and higher order correlation functions.

Two more comments about the two-point function of the scalar primary fields. If the fields have different scaling dimensions, $\Delta_1 \neq \Delta_2$, the two-point function vanishes

$$\langle \phi_1(x)\phi_2(y) \rangle = 0 . \quad (2.11)$$

This is clear from the cone perspective since we cannot construct the analogue of (2.1) is $\Delta_1 \neq \Delta_2$.

In a theory with several fields $\phi_i$ with same scaling dimension $\Delta$, the two-point function is

$$\langle \phi_i(x)\phi_j(y) \rangle = \frac{M_{ij}}{(x - y)^{2\Delta}} , \quad (2.12)$$

Here the matrix $M_{ij}$ will be positive-definite matrix for a unitary theory (as we will see in future lectures). This means that there exists a field basis such that $M_{ij}$ becomes diagonal

$$\langle \phi_i(x)\phi_j(y) \rangle = \frac{\delta_{ij}}{(x - y)^{2\Delta}} . \quad (2.13)$$

We will always assume that such a basis is chosen.

### 2.2 Primary scalar three point function

The three-point function of three primary scalar fields with scaling dimensions, $\Delta_1, \Delta_2, \Delta_3$ (could be equal or different) must have the following form on the cone

$$\langle \phi_1(X_1)\phi_2(X_2)\phi_3(X_3) \rangle = \frac{\text{const.}}{(X_1X_2)^{\alpha_{123}}(X_1X_3)^{\alpha_{132}}(X_2X_3)^{\alpha_{231}}} . \quad (2.14)$$
As in the two-point function, the above is the most general Lorentz invariant expression. To make it consistent with scaling, we should impose the constraints

\[
\begin{align*}
\alpha_{123} + \alpha_{132} &= \Delta_1 \\
\alpha_{123} + \alpha_{231} &= \Delta_2 \\
\alpha_{132} + \alpha_{231} &= \Delta_3 .
\end{align*}
\]

(2.15)

The above is a linear system with three equations for three parameters; it admits a unique solution given by

\[
\alpha_{ijk} = \frac{\Delta_i + \Delta_j - \Delta_k}{2} .
\]

(2.16)

Thus the three-point function is uniquely determined up to a constant. If we project (2.14) on the Euclidean section, we find

\[
\langle \phi_1(x_1)\phi_2(x_2)\phi_3(x_3) \rangle = \frac{\lambda_{123}}{(x_{12})^{2\alpha_{123}}(x_{13})^{2\alpha_{132}}(x_{23})^{2\alpha_{231}}} ,
\]

(2.17)

where \(\lambda_{123}\) a free parameter (which one can call a “coupling constant”), and

\[
x_{ij} = x_i - x_j .
\]

(2.18)

This formula was first derived by Polyakov in 1970 and is a remarkable result. One can say that it gave birth to CFT. To understand its significance, we should compare it with infinitely many functional forms which would be allowed if we imposed only SI:

\[
\sum \text{const.} \left( \frac{1}{|x_{12}|^a |x_{13}|^b |x_{23}|^c} \right), \quad a + b + c = \Delta_1 + \Delta_2 + \Delta_3 ,
\]

(2.19)

whereas there is only one term consistent with CI.

As a concrete example, the three-point function \(\langle \sigma(x)\sigma(y)\epsilon(z) \rangle\) for two spins and energy in the 2-dimensional Ising model at the critical point can be extracted from the exact Onsager’s lattice solution. That it agrees with Polyakov’s formula is an evidence for the CI of the critical 2D Ising model.

### 2.3 Four point function

We now move on to the four-point function. For simplicity we will only consider four identical fields. Requiring consistency under Lorentz transformations and scaling, we get on the cone

\[
\langle \phi(X_1)\phi(X_2)\phi(X_3)\phi(X_4) \rangle = \frac{1}{(X_1 \cdot X_2)^{\Delta}(X_3 \cdot X_4)^{\Delta}} f(u, v) ,
\]

(2.20)

Here \(u\) and \(v\) are conformally invariant cross-ratios which on the light-cone are given by Lorentz-invariant expressions

\[
u = \frac{(X_1 \cdot X_2)(X_3 \cdot X_4)}{(X_1 \cdot X_3)(X_2 \cdot X_4)} , \quad \text{and} \quad \nu = u|_{2+4} = \frac{(X_1 \cdot X_4)(X_2 \cdot X_3)}{(X_1 \cdot X_3)(X_2 \cdot X_4)} .
\]

(2.21)
Notice that they have scaling zero in every variable. Since the first term in the RHS takes care of the scaling, any function $f(u, v)$ of $u$ and $v$ can appear as a factor.

Now as we project to the Euclidean section, $X_1 \cdot X_2, X_3 \cdot X_4$ project to $x^2_{12}, x^2_{34}$ respectively, while $u$ and $v$ become

$$u = \frac{x^2_{12}x^2_{34}}{x^2_{13}x^2_{24}}, \quad \text{and} \quad v = u|_{2 \leftrightarrow 4} = \frac{x^2_{14}x^2_{23}}{x^2_{13}x^2_{24}}. \quad (2.22)$$

That the four-point function must be given by a simple expression times a function of the conformally invariant cross-ratios is an enormous reduction of the functional freedom, although not as large as for the three point functions where the functional form was completely fixed.

We will later see that $f(u, v)$ is not an independent quantity but is related in a non-trivial way to the three-point function. But this will require dynamics, while here we are just doing kinematics.

For the moment let us notice a functional constraint on $f(u, v)$ which comes from the crossing symmetry of the four-point function. We saw that

$$\langle \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4) \rangle = \frac{1}{x_{12}^{2\Delta}x_{34}^{2\Delta}} f(u, v). \quad (2.23)$$

In the above expression there was no particular reason to group the coordinates that way. If for example we interchange 2 $\leftrightarrow$ 4, we get

$$\langle \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4) \rangle = \frac{1}{x_{14}^{2\Delta}x_{23}^{2\Delta}} f(\tilde{u}, \tilde{v}), \quad (2.24)$$

where now $f(\tilde{u}, \tilde{v})$ depends on the conformally invariant cross sections calculated with the interchanged indices. In this case, $2 \leftrightarrow 4$ simply means

$$\tilde{u} = v, \quad \tilde{v} = u. \quad (2.25)$$

Notice that the same function $f$ appears in (2.23) and (2.24), since the four point function is totally symmetric under permutations. Moreover, (2.23) and (2.24) must agree:

$$\frac{1}{x_{12}^{2\Delta}x_{34}^{2\Delta}} f(u, v) = \frac{1}{x_{14}^{2\Delta}x_{23}^{2\Delta}} f(v, u) \quad (2.26)$$

Multiplying by $x_{14}^{2\Delta}x_{23}^{2\Delta}$, we find that $f(u, v)$ must satisfy:

$$\left(\frac{v}{u}\right)^\Delta f(u, v) = f(v, u). \quad (2.27)$$

This constraint will play an important role in the last lecture.
3 Fields with spin

3.1 Extending null cone formalism

So far we only talked about scalar primaries. Let us now consider primaries with spin.

We will consider symmetric traceless primary fields living on the $D$ dimensional space. We will put such a field in correspondence with a fields which lives on the light-cone and is also symmetric and traceless:

$$\phi_{\mu\nu\lambda\ldots}(x) \leftrightarrow \phi_{MNL\ldots}(X).$$

(3.1)

We notice that the fields on the light-cone have more components that the $D$ dimensional ones. For this correspondence to be useful, we have to eliminate the extra components. Let’s first of all impose transversality of the null cone fields

$$X^M \phi_{MNL\ldots}(X) = 0.$$  

(3.2)

This condition eliminates one component per index. We will see below how the remaining one is dealt with.

Then we define $\phi_{\mu\nu\lambda\ldots}(x)$ to be related to $\phi_{MNL\ldots}(X)$ by projection on the Euclidean section

$$\phi_{\mu\nu\lambda\ldots}(x) = \phi_{MNL\ldots}(X) \frac{\partial X^M}{\partial x^\mu} \frac{\partial X^N}{\partial x^\nu} \ldots,$$

(3.3)

where $X^M = (1, x^2, x^\mu)$ is the parametrization of the section, so

$$\frac{\partial X^M}{\partial x^\nu} = (0, 2x_\nu, \delta^M_\nu).$$  

(3.4)

Notice that this rule preserves the tracelessness condition: if we start from a traceless tensor, we will end up with a traceless tensor as well. Indeed, to compute the trace of $\phi_{MN\ldots}$ we have to evaluate the contraction

$$\delta^{\mu\nu} \frac{\partial X^M}{\partial x^\mu} \frac{\partial X^N}{\partial x^\nu},$$

(3.5)

which can be shown to be equal to

$$\eta^{MN} + X^M \cdot K^N + X^N \cdot K^M,$$

(3.6)

with $K_M = (0, 2, 0)$ an auxiliary vector. Contracted with $\phi_{MN\ldots}$, it will vanish by tracelessness and transversality.

Notice also that anything proportional to $X^M$ projects to zero, since

$$X^2 = 0 \Rightarrow X^M \frac{\partial X^M}{\partial x^\mu} = 0.$$  

(3.7)

1Primaries in other representations of $SO(D)$, like antisymmetric tensors of fermions, can also be considered
This means that $\phi_{MNL\ldots}$ is defined up to adding an arbitrary tensor proportional to $X^M$. This “gauge invariance” reduces the number of degrees of freedom to the needed one.

Let’s discuss the transformation properties. Under an $SO(D+1,1)$ transformation, the field on the null cone transforms in the standard Lorentz invariant way:

$$\tilde{\phi}_{MNL\ldots}(X') = \Lambda^M_N \Lambda^{N'}_{N'} \cdots \phi_{M'N'L'\ldots}(X),$$

(3.8)

Just like for primary scalars, we will impose that the null cone fields are homogeneous in $X$:

$$\phi_{\ldots}(\lambda X) = \lambda^{-\Delta} \phi_{\ldots}(X),$$

(3.9)

We claim that the resulting transformations for the fields on the section is what we need:

$$\tilde{\phi}_{\ldots}(x') = \frac{1}{b(x)^{\Delta}} O^{\nu}_{\mu}(x) \cdots \phi_{\ldots}(x).$$

(3.10)

Here the line element transforms as

$$dx' = b(x)O(x).dx$$

(3.11)

To show that (3.10) is true, it’s enough to show that (we consider spin 1 case for simplicity)

$$\tilde{\phi}(x').dx' = \frac{1}{b(x)^{\Delta-1}} \phi(x).dx$$

(3.12)

Now, the projection rule implies that

$$\phi(x).dx = \phi(X).dX$$

(3.13)

When $X \to \Lambda X$ the scalar product $\phi(X).dX$ is preserved:

$$\phi(Y).dY = \phi(X).dX, \quad Y = \Lambda X$$

(3.14)

To get from $Y$ back into the section we have to rescale: $X' = bY$. When we do it $\phi(Y)$ simply rescales. $dY$ rescales plus gets a contribution proportional to $Y$ if $b$ is not a constant. This extra contribution vanishes when contracted with $\phi(Y)$ because of transversality. The end result is exactly (3.12)

### 3.2 Two point function

Let us now see the consequences. Consider the two-point function of a vector field. On the cone we have:

$$\langle \phi_M(X)\phi_N(Y) \rangle = \frac{\eta_{MN} + \alpha \frac{Y_M X_N}{XY}}{(XY)^\Delta},$$

(3.15)

where we once again considered the most general Lorentz invariant form consistent with scaling. Notice that we don’t write terms proportional to $X_M$ or $Y_N$, since they anyway project to zero.
We have to impose transversality which fixes the value of the constant $\alpha$

$$X^M(\ ) = Y^N(\ ) = 0 \Rightarrow \alpha = -1 . \quad (3.16)$$

Projecting the two-point function in the physical space, we find:

$$\eta_{MN} \rightarrow \delta_{\mu\nu}$$
$$Y_M \rightarrow -x_\mu + y_\mu$$
$$X_N \rightarrow x_\nu - y_\nu$$
$$X \cdot Y \rightarrow -\frac{1}{2} (x - y)^2 ,$$

therefore

$$\eta_{MN} - \frac{Y_M Y_N}{(XY)^\Delta} \rightarrow I_{\mu\nu}(x - y) \frac{(x - y)^2}{(x - y)^{2\Delta}} , \quad (3.17)$$

with

$$I_{\mu\nu}(x) = \delta_{\mu\nu} - \frac{2x_\mu x_\nu}{x^2} . \quad (3.18)$$

Notice that in the above expression it is CI that fixes the relative coefficient $-2$ between the two terms. If we had SI only, this coefficient would be free.

[It would not be so easy to check that the found two point function transforms correctly under CT without using the cone. As usual, it would be sufficient to check how it transforms correctly under inversion. This in turn would be equivalent to the identify

$$I(x)I(x - y)I(y) = I(x' - y') , \quad (3.20)$$

with $x' = x/x^2$. One can check this by an explicit computation, expanding throughout, but it looks completely accidental this way.]

Similarly, the two point function can be computed for higher spin primary fields. Interestingly, one discovers that, apart from $I_{\mu\nu}$, no new conformally covariant tensors appear which connect two different points. All the two point functions are made of $I_{\mu\nu}$’s connecting different points, and $\delta_{\mu\nu}$’s if the indices $\mu, \nu$ are associated with the same point. For example, the two-point function for a symmetric traceless field will be

$$\langle \phi_{\mu\nu}(x)\phi_{\lambda\sigma}(y) \rangle = \frac{1}{|x - y|^{2\Delta}}[I_{\mu\lambda}(x - y)I_{\nu\sigma}(x - y) + (\mu \leftrightarrow \nu) + \alpha\delta_{\mu\nu}\delta_{\lambda\sigma}] , \quad (3.21)$$

where the term with the delta functions (transforming correctly under CT since both indices get multiplied by the same $O_\mu$) was inserted in order to satisfy the tracelessness condition, which fixes the value of $\alpha$:

$$\alpha = -\frac{2}{D} . \quad (3.22)$$

To summarize, the two-point functions are completely fixed for higher spin primaries just like for the scalar.
3.3 Remark about inversion

In “pedestrian” CFT calculations, not based on the projective null cone formalism, one often checks invariance under the inversion rather than under SCT. However, as we mentioned, inversion is not on the connected part of the conformal group. So is assuming invariance under the inversion an extra assumption?

On the cone, the inversion corresponds to the transformation

\[ X^{D+1} \rightarrow -X^{D+1} \]

\[ X^\pm = X^{D+2} \pm X^{D+1} \rightarrow X^+ \leftrightarrow X^- . \]  

(3.23)

Indeed, if we start with a point \( X^M = (1, x^2, x^\mu) \) on the Euclidean section, we end up with

\( (1, x^2, x^\mu) \rightarrow (x^2, 1, x^\mu) \) rescale with \( x^2 \rightarrow (1, 1/x^2, x^\mu/x^2) \),

(3.24)

which has inversion in the last component.

Notice that the transformation \( X^{D+1} \rightarrow -X^{D+1} \) is in \( O(D+1, 1) \) but not in \( SO \), i.e. it is not in the connected component.

Another transformation in the same class is a simple spatial reflection (parity transformation)

\[ X^1 \rightarrow -X^1 . \]  

(3.25)

The two discrete symmetries parity and inversion are conjugate by \( SO(D+1, 1) \), which means that if we add to this group one of those, we get the same group. This implies that a CFT invariant under parity will be invariant under inversion and vice versa.

There are CFTs which break parity (and inversion). Correlators in those theories, lifted to the null cone, will involve the \( D + 2 \) dimensional \( \epsilon \)-tensor, or \( \Gamma_{D+3} \) (the analog of \( \gamma_5 \) matrix) for fermions if \( D + 2 \) is even. Since we only considered scalars and symmetric tensors, these structures did not occur.

3.4 Remark on conservation

We have seen that for spin-1 and spin-2 primary fields, the form of the two-point correlation functions is fixed by CI in terms of just one parameter: the dimension of the field. Canonical dimensions

\[ \Delta = D - 1 , \quad \text{for} \quad l = 1 , \quad \Delta = D , \quad \text{for} \quad l = 2 . \]  

(3.26)

would correspond to the conserved currents and the stress tensor. We expect their two-point functions to be conserved objects. This should happen automatically since there is nothing to be adjusted. And indeed one can check that this is true. E.g. for the currents

\[ \partial^\mu I_{\mu\nu}(x) = 0 , \quad \text{for} \quad \Delta = D - 1 , \]  

(3.27)
Notice that the null cone formalism is simply a way to compute constraints imposed by CI. For example, current and stress tensor conservation may be more convenient to check in the physical space rather than on the null cone. There is no reason to insist in doing everything on the null cone. The two points of view - null cone and physical space - can be used interchangeably depending what one wants to compute.

### 3.5 Scalar-scalar-spin \( l \)

The last correlator that we will study in this lecture is the three-point function of two scalars and one spin \( l \) operator. Start with spin one. On the null cone we will have

\[
\langle \phi_1(X)\phi_2(Y)\phi_3(Z) \rangle = \text{scalar factor} \times (\text{tensor structure})_M ,
\]

(3.28)

The scalar factor will be the same as for the scalars

\[
\frac{\text{const.}}{(XY)^{\alpha_{123}}(YZ)^{\alpha_{231}}(XZ)^{\alpha_{132}}} ,
\]

(3.29)

where the powers are fixed by the dimensions of the fields in order to get the correct scaling. The tensor structure must then have scaling 0 in all variables, and will also have to be transverse \( Z^M(\ )_M = 0 \). Moreover we don’t need to include a term proportional to \( Z_M \) since it will project to zero. It’s then easy to see that the tensor part must be equal to

\[
\frac{(YZ)X_M - (XZ)Y_M}{(XZ)^{1/2}(XY)^{1/2}(YZ)^{1/2}} ,
\]

(3.30)

where the minus sign was fixed from the transversality constraint. We now have to project the tensor part into the physical space, which means we have to multiply by \( \partial Z^M/\partial z^\mu \). We find that

\[
X_M \rightarrow (x - z)_\mu \quad \text{and} \quad Y_M \rightarrow (y - z)_\mu ,
\]

(3.31)

therefore the expression projects into

\[
\frac{(x - z)_\mu |y - z|^2 - (y - z)_\mu |x - z|^2}{|x - z||y - z||x - y|} ,
\]

(3.32)

or in a nicer form

\[
\frac{|y - z||x - z|}{|x - y|} \left( \frac{(x - z)_\mu}{|x - z|^2} - \frac{(y - z)_\mu}{|y - z|^2} \right) \equiv R_\mu(x, y|z) .
\]

(3.33)

The quantity \( R_\mu \) transforms correctly under CT, so for a symmetric traceless field with more indices. Again it turns out that this is the only indexed object for three points with this property (\( I_{\mu\nu} \) is not useful here since it has two indices at different points). For spin \( l \) fields the above is generalized into

\[
\langle \phi_1(x)\phi_2(y)\phi_{3\mu\nu\lambda\cdots}(z) \rangle \propto \text{scalar part} \times (R_\mu R_\nu R_\lambda \cdots - \text{traces}) .
\]

(3.34)
We see that the three-point function is again completely fixed up to an arbitrary constant. Notice that we cannot use

Suppose that we now look at the three-point functions of two scalars and the current \( J_\mu \) and the stress tensor \( T_{\mu\nu} \)

\[
\langle \phi_1(x) \phi_2(y) J_\mu(z) \rangle, \quad \langle \phi_1(x) \phi_2(y) T_{\mu\nu}(z) \rangle. \quad (3.35)
\]

What if we impose here the conservation condition? For two point functions conservation was automatic, but here it is not so. In fact, these three point functions are conserved if and only if the scalars have equal dimensions, \( \Delta_1 = \Delta_2 \). Intuitively this happens because the three-point function must satisfy the Ward identities, which relate it to the two-point function. As we have seen, the two-point function is non-zero if and only if \( \Delta_1 = \Delta_2 \). The conclusion is that the coupling of the stress tensor and conserved currents to two scalar primaries of unequal dimensions must vanish.

### 4 An elementary property of CTs

The following simple property of conformal transformations will be needed in the future: they map circles to circles (including straight lines which are circles with infinite radius).

To prove it, consider a circle in \( \mathbb{R}^D \), given by

\[
|x - x_0|^2 = R_0^2 \iff x^2 - 2x.x_0 + (x_0^2 - R_0^2) = 0 \quad (4.1)
\]

It’s enough to show that the image under inversion is a circle, i.e.

\[
|x/x^2 - x_1|^2 = R_1^2 \iff x^2(x_1^2 - R_1^2) - 2x.x_1 + 1 = 0 \quad (4.2)
\]

This is true if we fix \( x_1 \) and \( R_1 \) from the equations

\[
(x_1^2 - R_1^2)^{-1} = x_0^2 - R_0^2, \quad x_1/(x_1^2 - R_1^2) = x_0 \quad (4.3)
\]

Q.E.D.

This property of CT is well known in classical geometry. As an example, consider a problem which becomes very easy if one uses it and would be tricky to solve otherwise: Given a circle \( \gamma \) and two point \( A, B \) outside of it, construct a circle tangent to \( \gamma \) that passes through \( A \) and \( B \).

Idea of solution: Pick a points on the circle \( \gamma \) and apply an inversion with respect to that point. The circle is now mapped onto a straight line. In these coordinates, the problem can be seen to reduce to solving a quadratic equation. Then map back.

In this lecture we learned how to dominate the kinematics of the conformal group. Even though it gets us a long way, by itself it is not enough to solve a theory. In the next lecture we will move to more dynamical issues, in particular the Operator Product Expansion (OPE).
Literature

Projective null cone idea is due to Dirac, and was used by Mack and Salam, Ferrara et al, Siegel and others. More recently it was used by Cornalba et al in its essentially modern form, and then rediscovered by Weinberg who nicely wrote it up:


This formalism is also developed further in:


For the pedestrian approach to conformal correlators see


The classic paper by Polyakov which gave birth to CFTs: